Semantic Foundations of Equality Saturation

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[1] University of Washington [2] University of California, Los Angeles

Equality Saturation:
A general framework for program optimization

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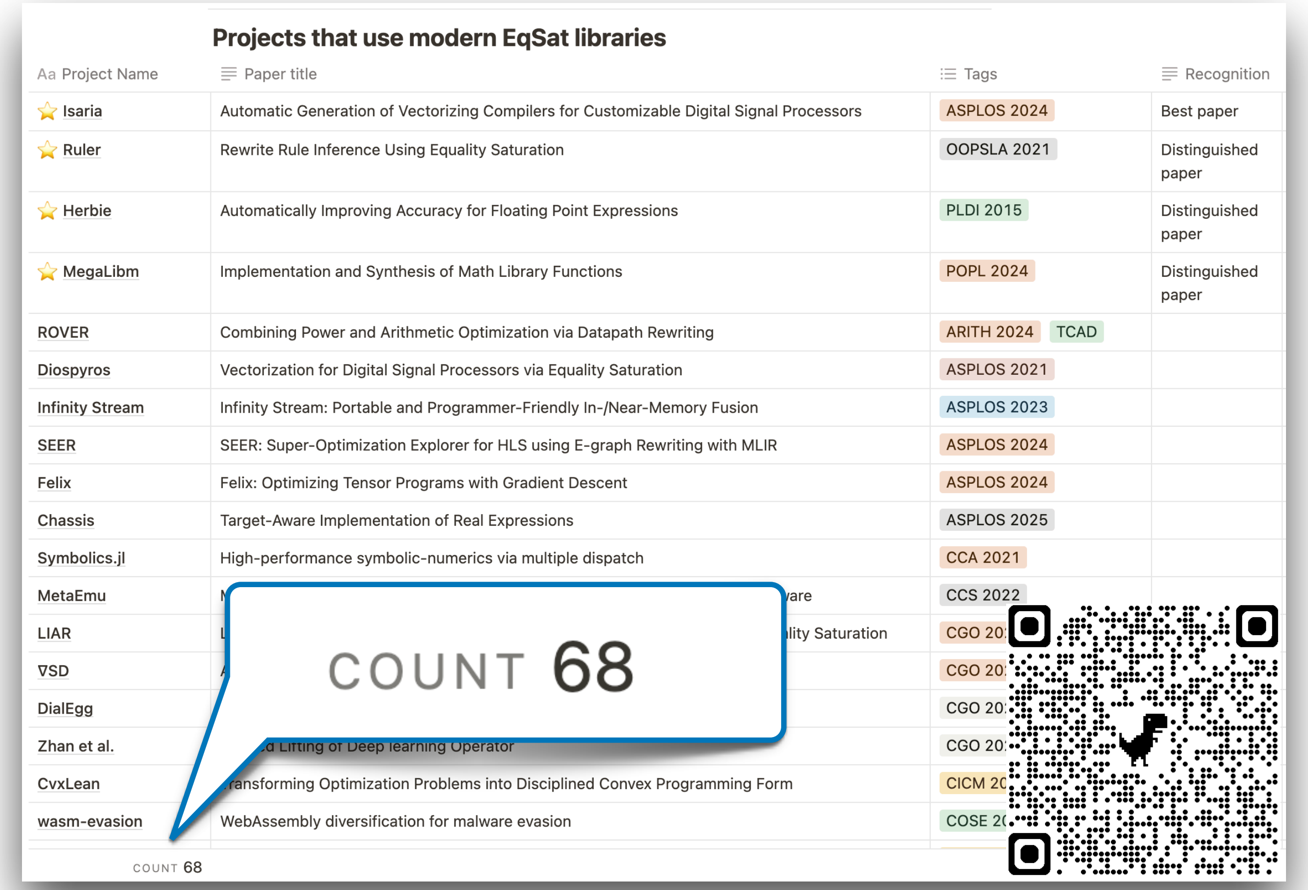

Projects that use modern EqSat libraries

Project Name	Paper title	Tags	Recognition
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wasm-evasion	WebAssembly diversification for malware evasion	COSE 20…	

COUNT 68

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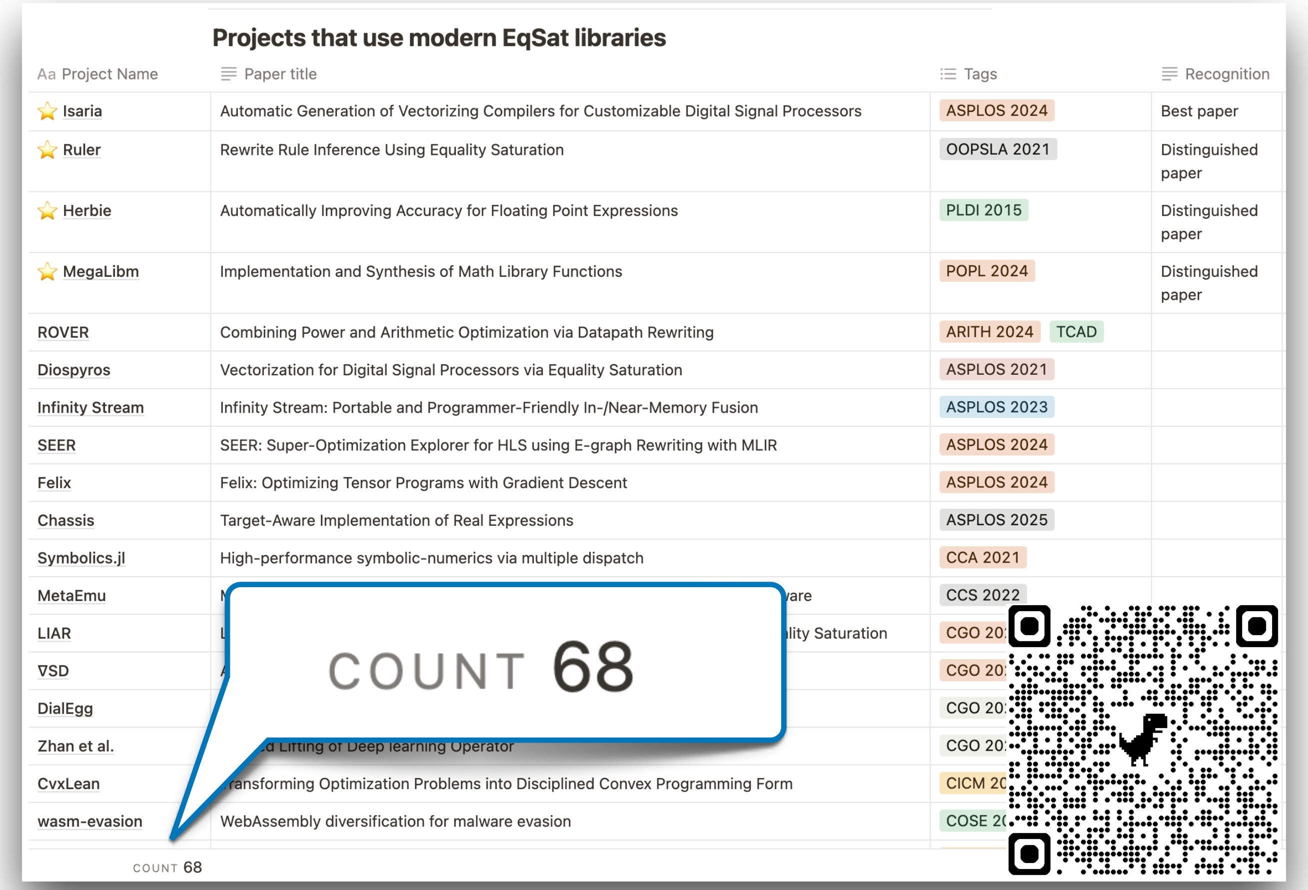

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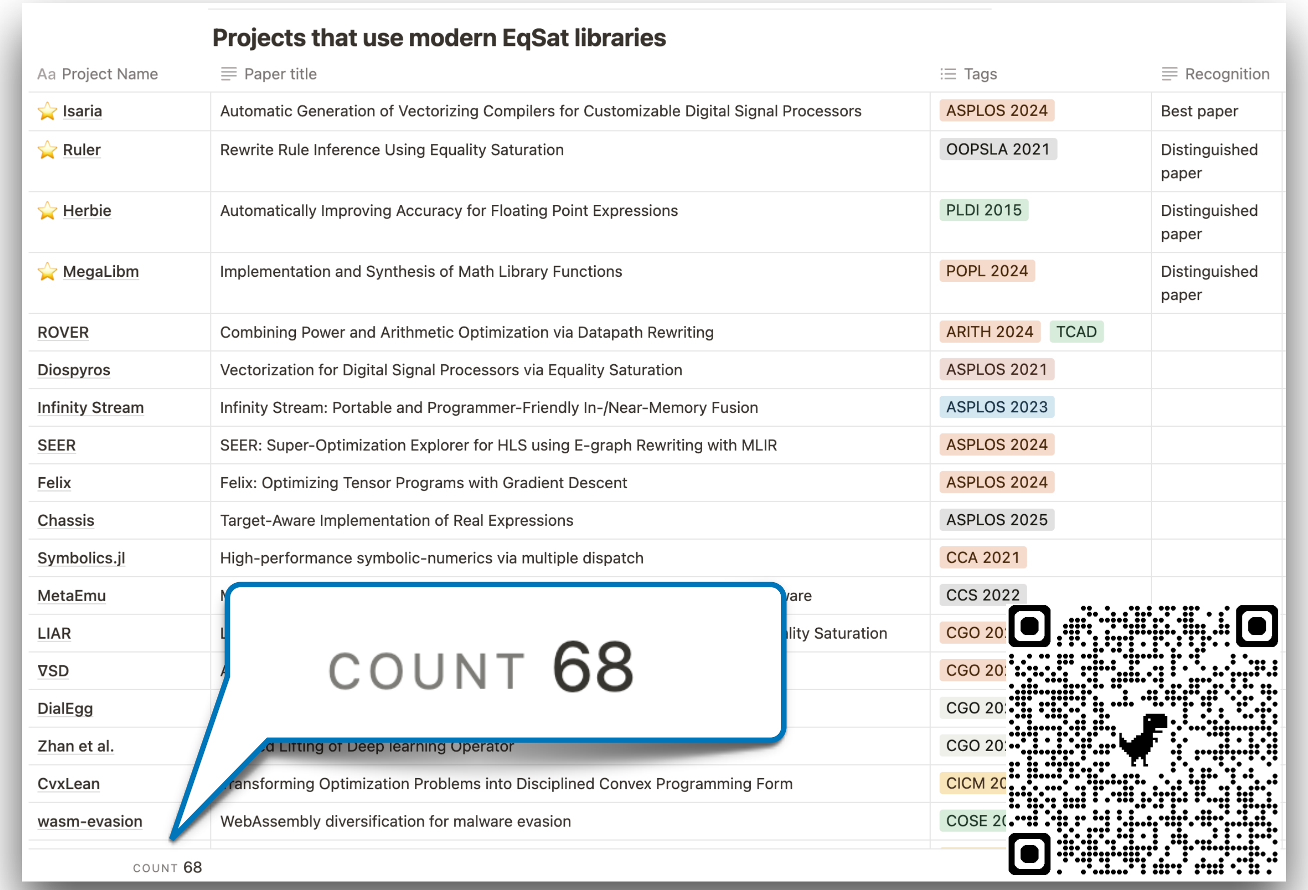

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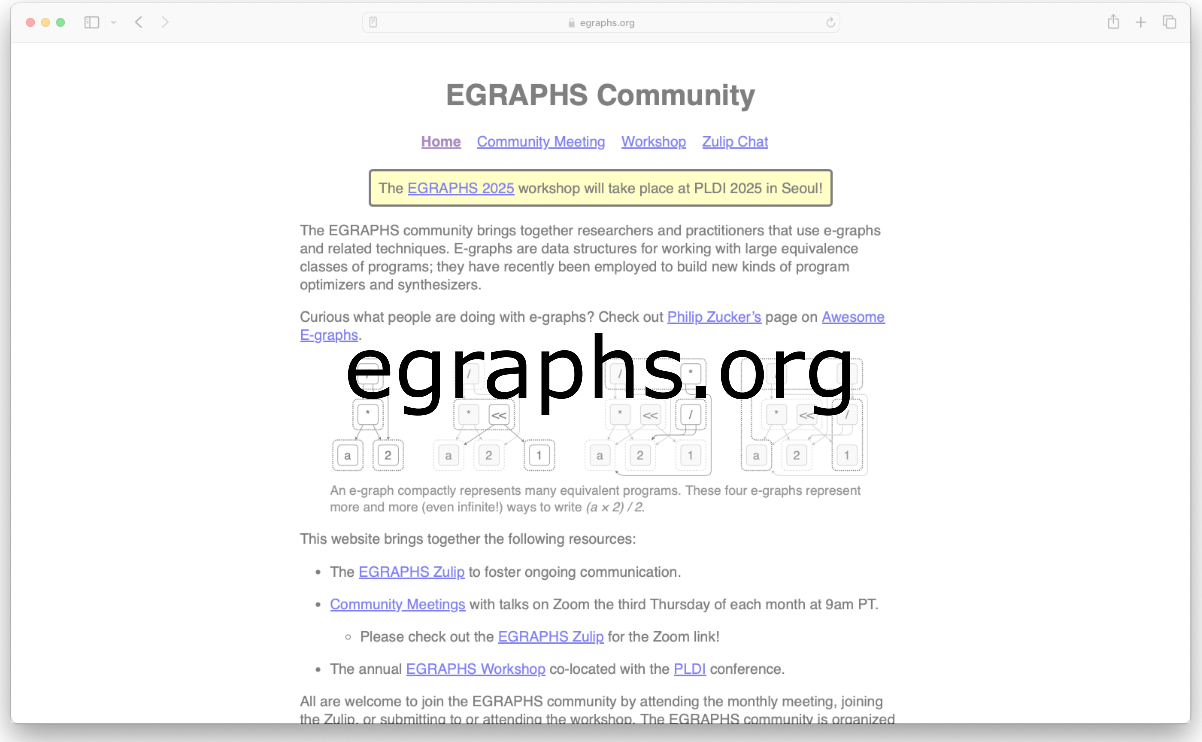

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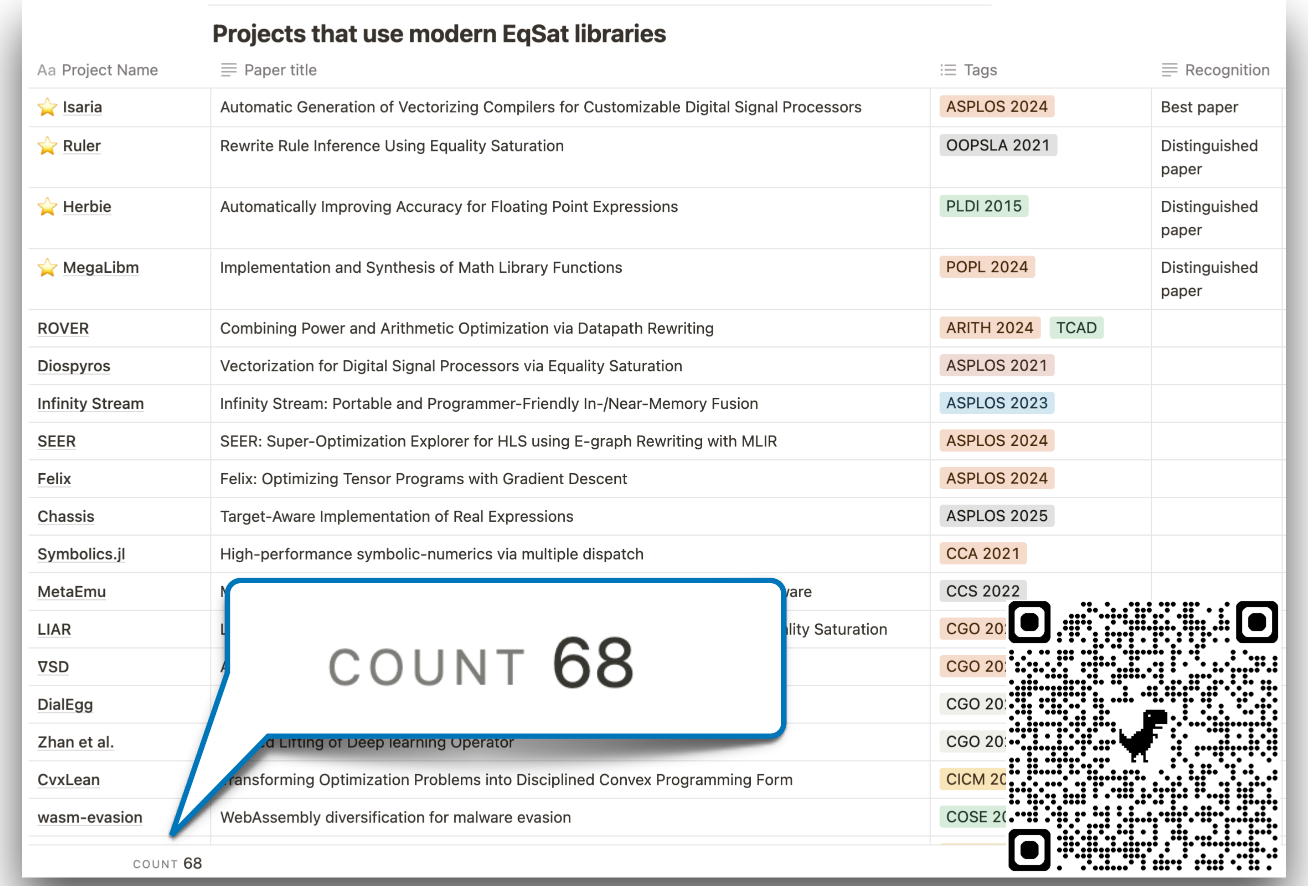

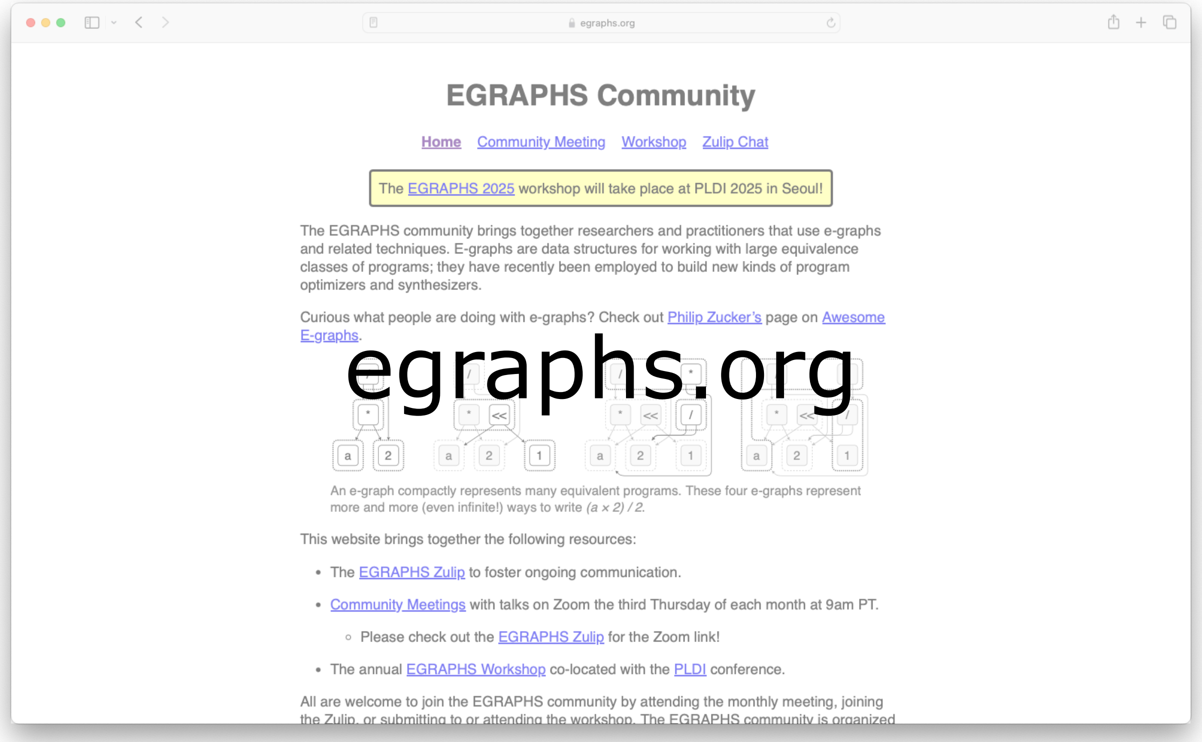

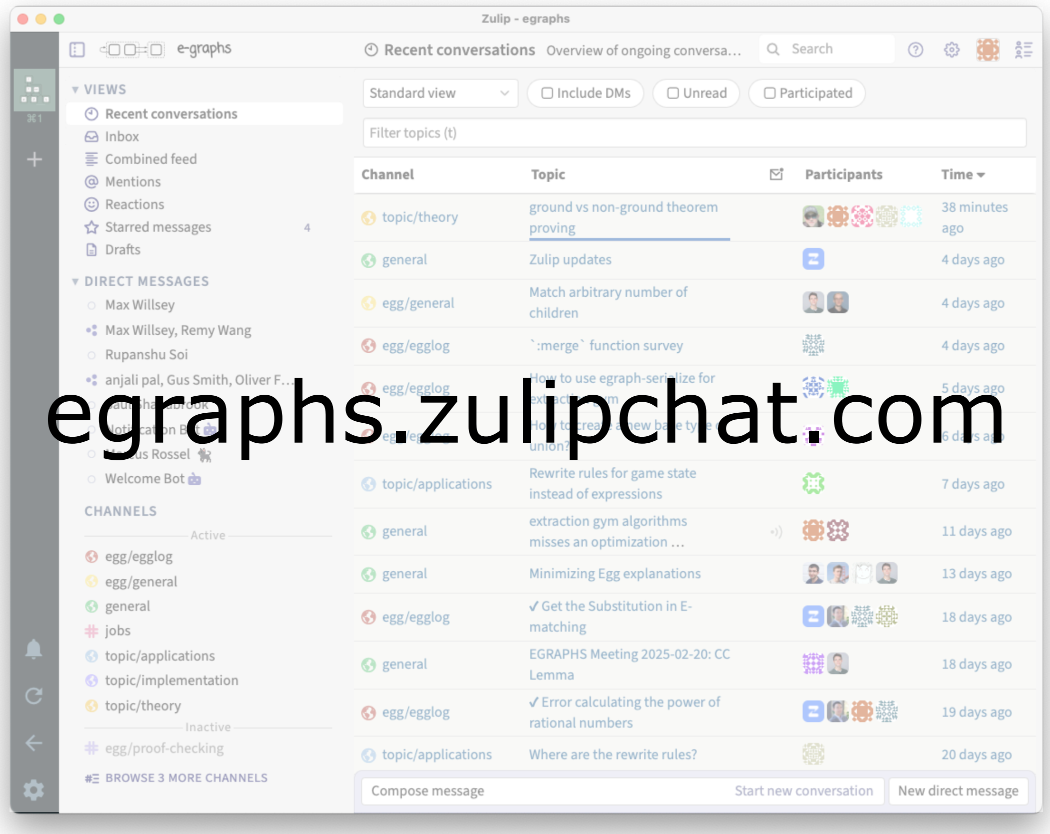

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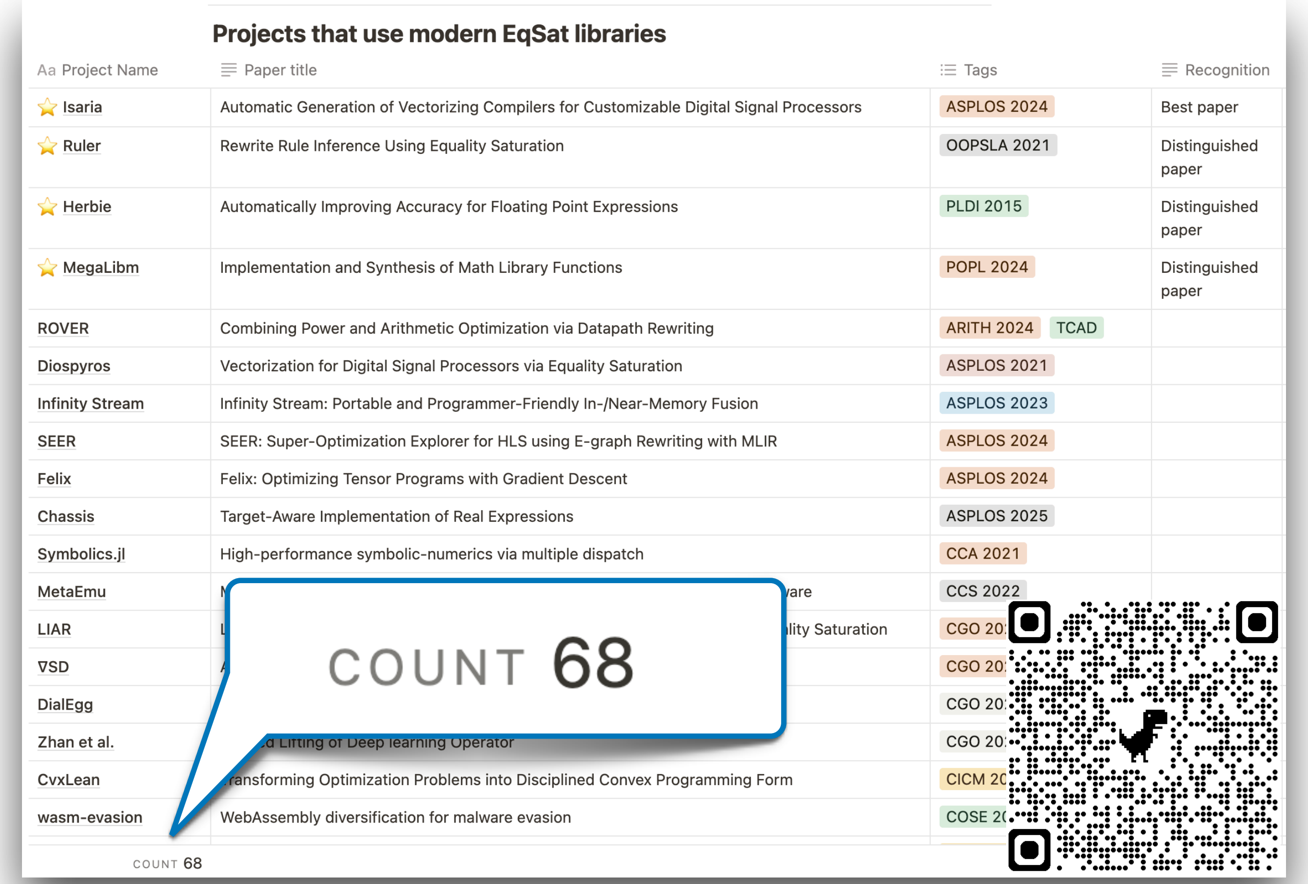

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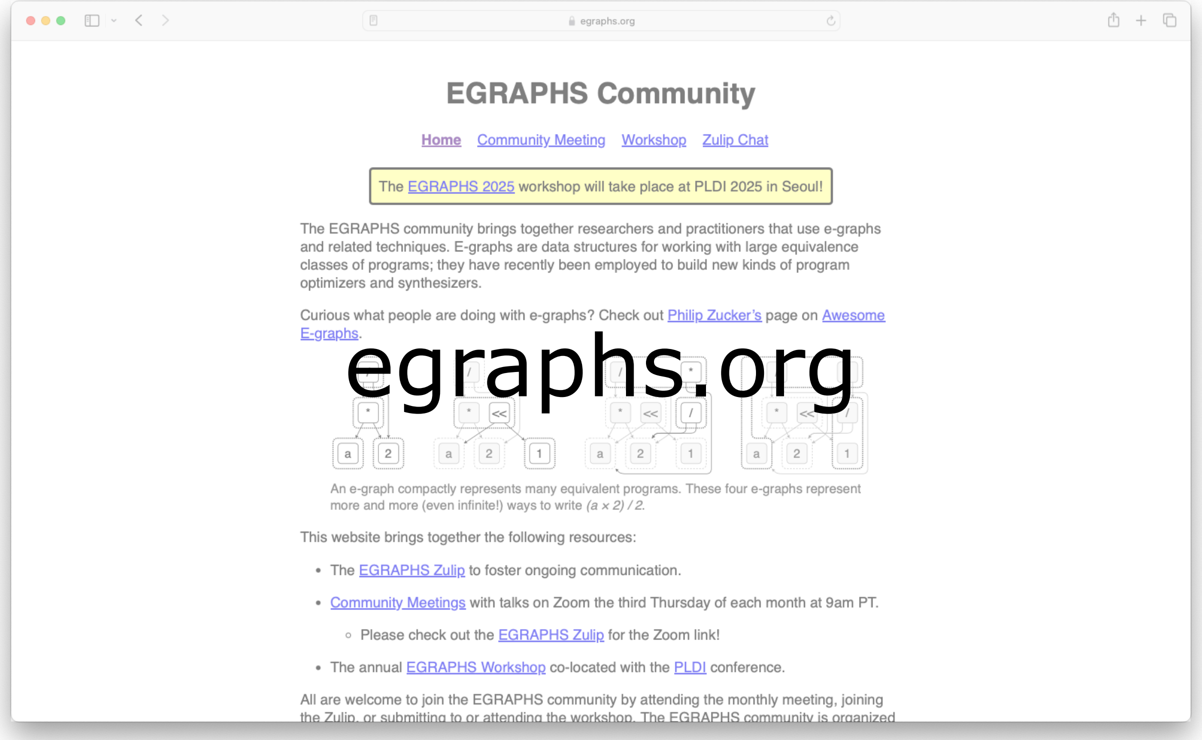

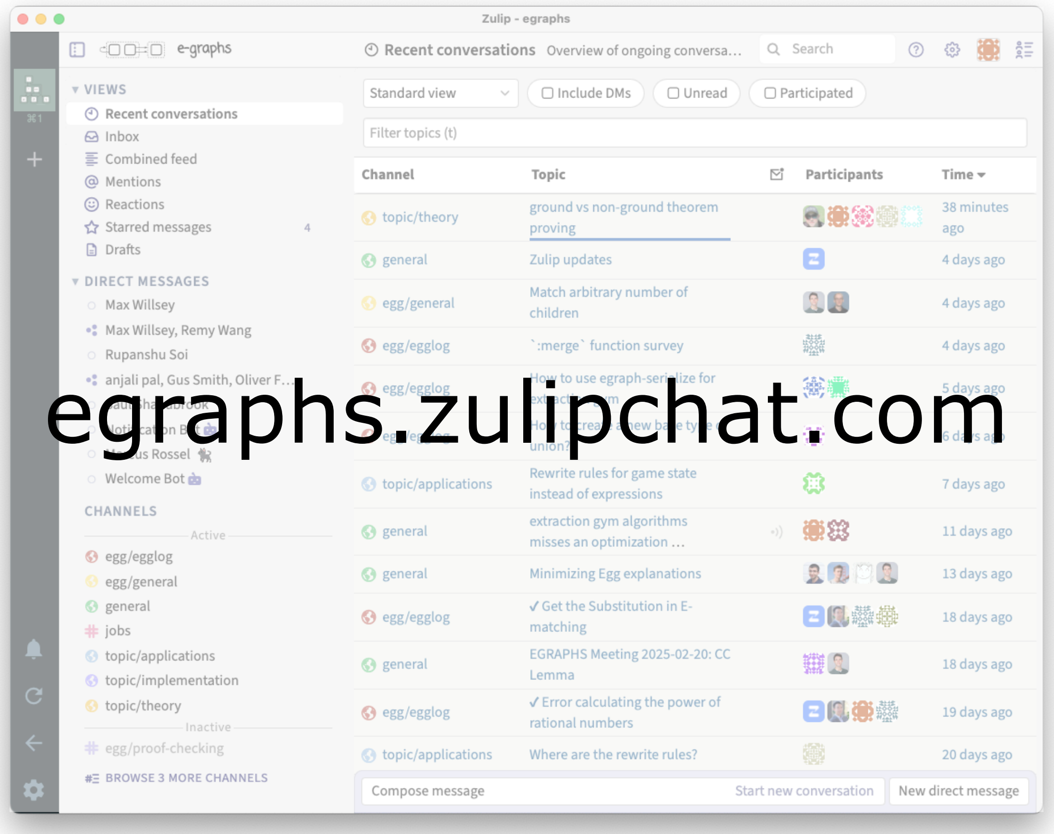

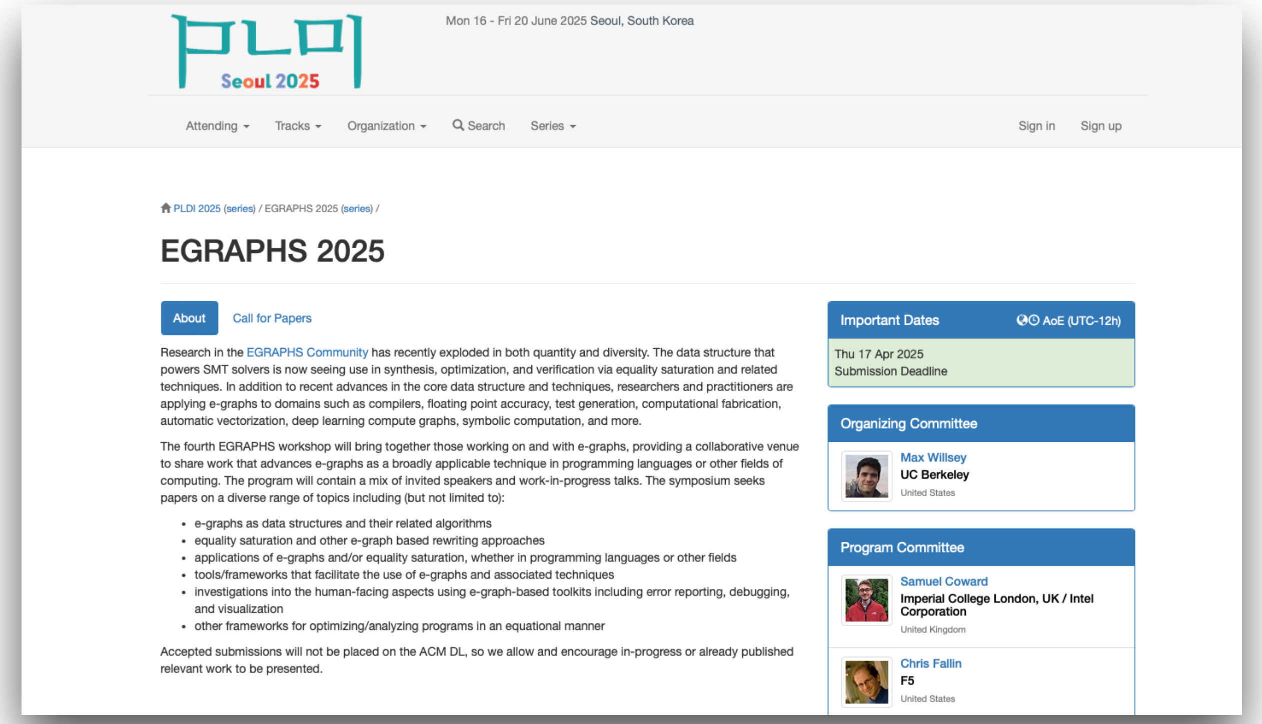

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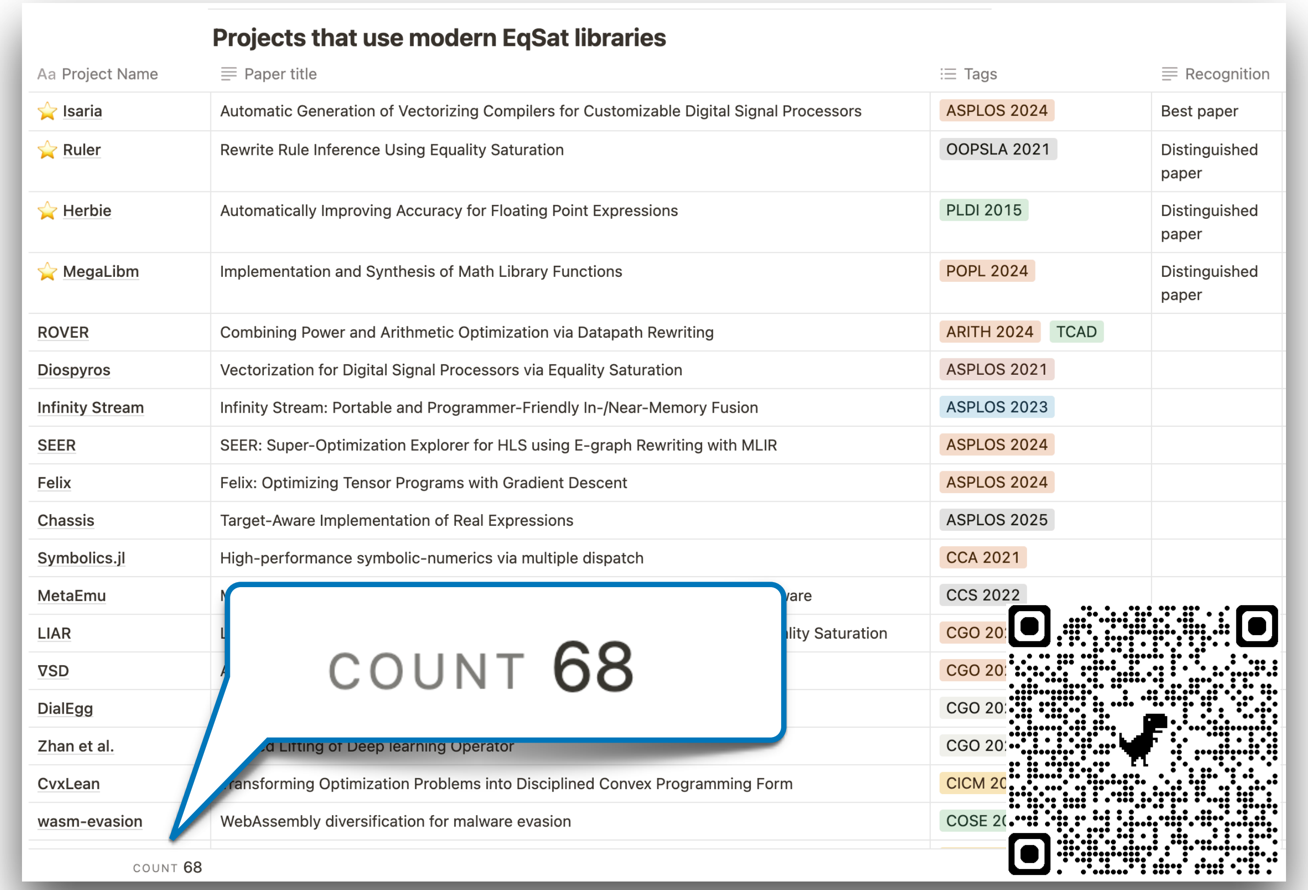
<u>Database theory community can help!</u>



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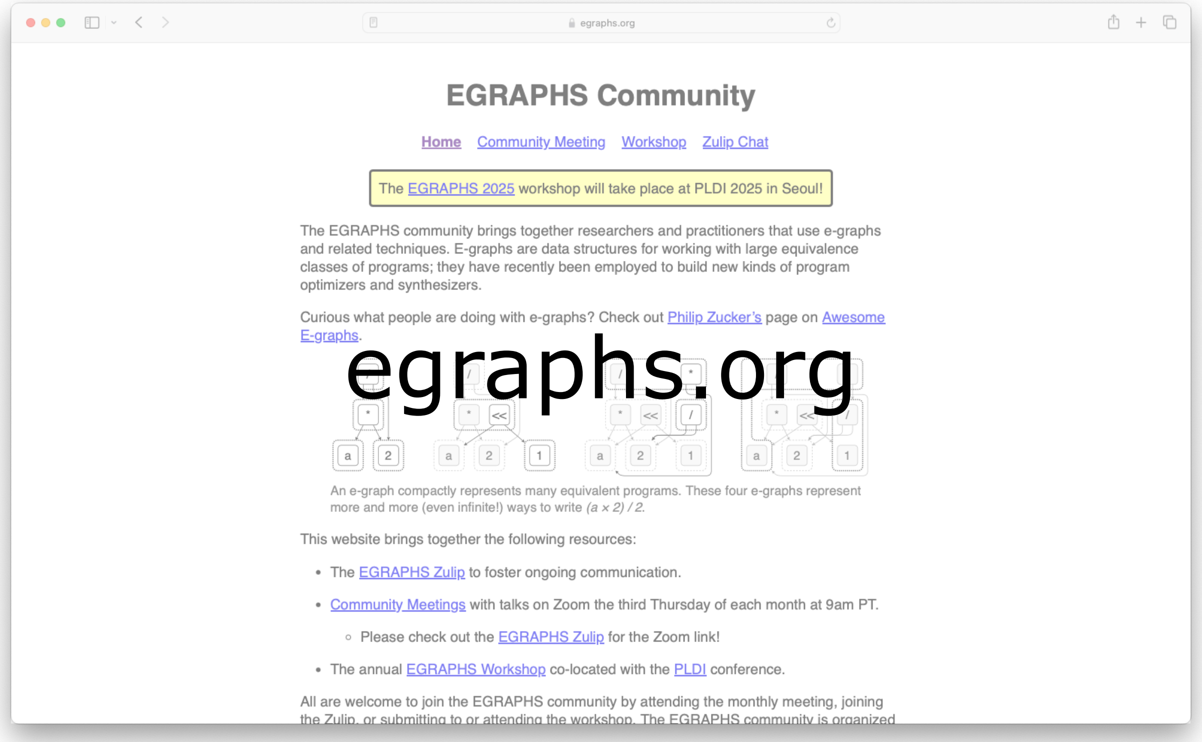

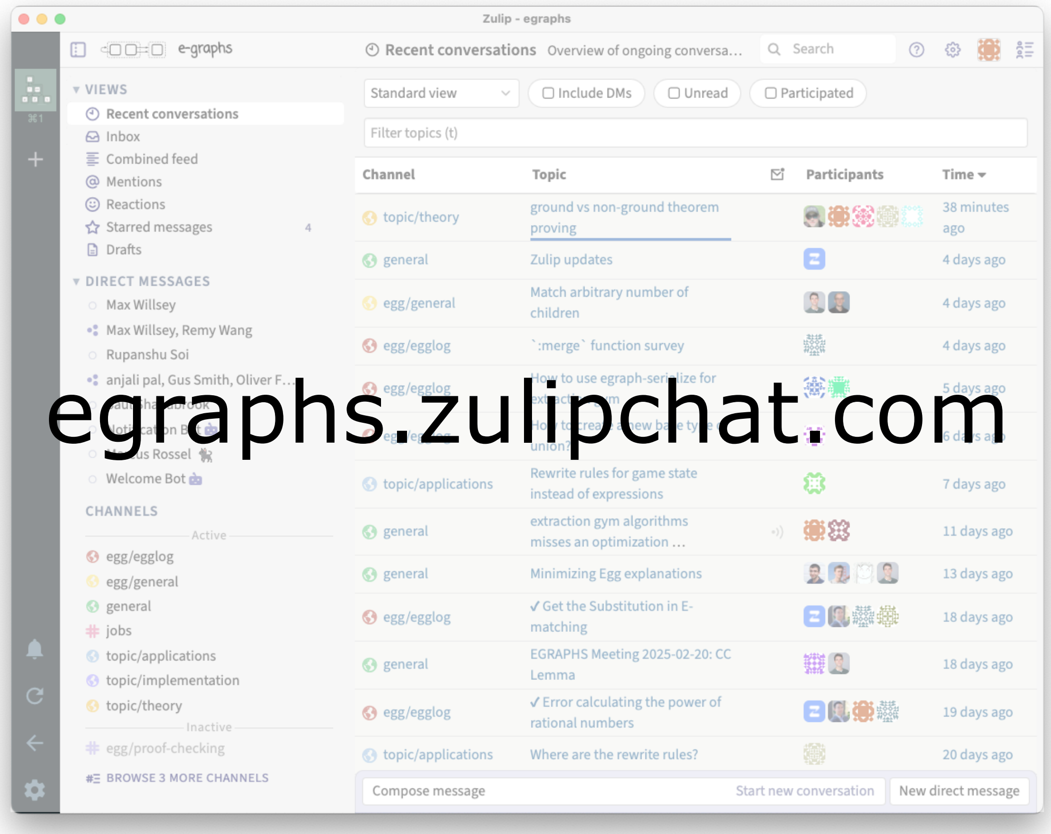

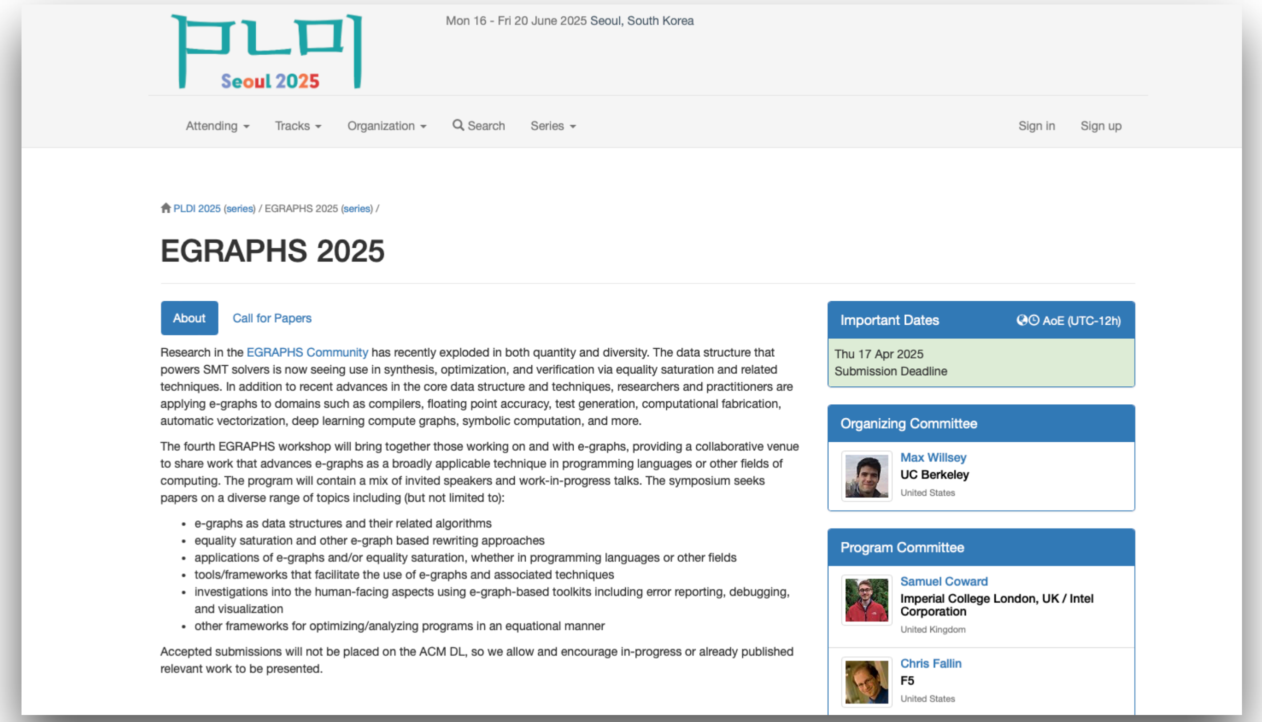

This paper

Defines a rigorous semantics to Equality Saturation (EqSat).

Studies EqSat in relationship to Term Rewriting and the Chase.

Proves the undecidability of EqSat termination in three cases.

The word problem and term rewriting

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Yes.

$$\begin{aligned}(a^{-1} \cdot b^{-1})^{-1} &\to (b^{-1})^{-1} \cdot (a^{-1})^{-1} \\ &\to b \cdot (a^{-1})^{-1} \\ &\to b \cdot a\end{aligned}$$

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Ruled-based program optimization

$$x \div x \to 1 \qquad (x \times y) \div z \to x \times (y \div z)$$

$$x \times 1 \to x \qquad x \times 2 \to x \ll 1$$

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$$(a \times 2) \div 2 \longrightarrow a \times (2 \div 2) \longrightarrow a \times 1 \longrightarrow a$$

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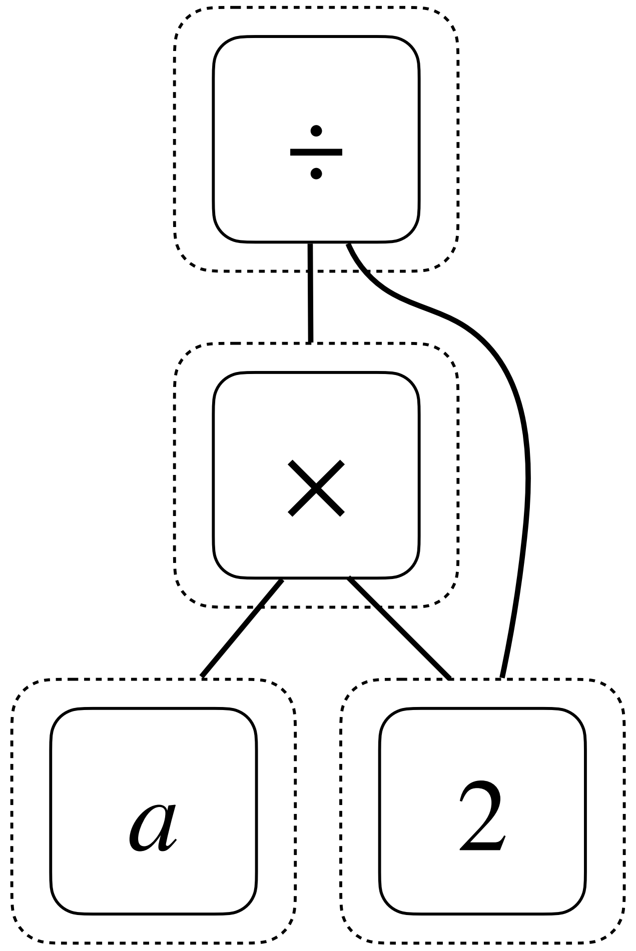
Equality Saturation is an algorithm to efficiently explore the program space defined by rules.

E-graphs and Equality Saturation



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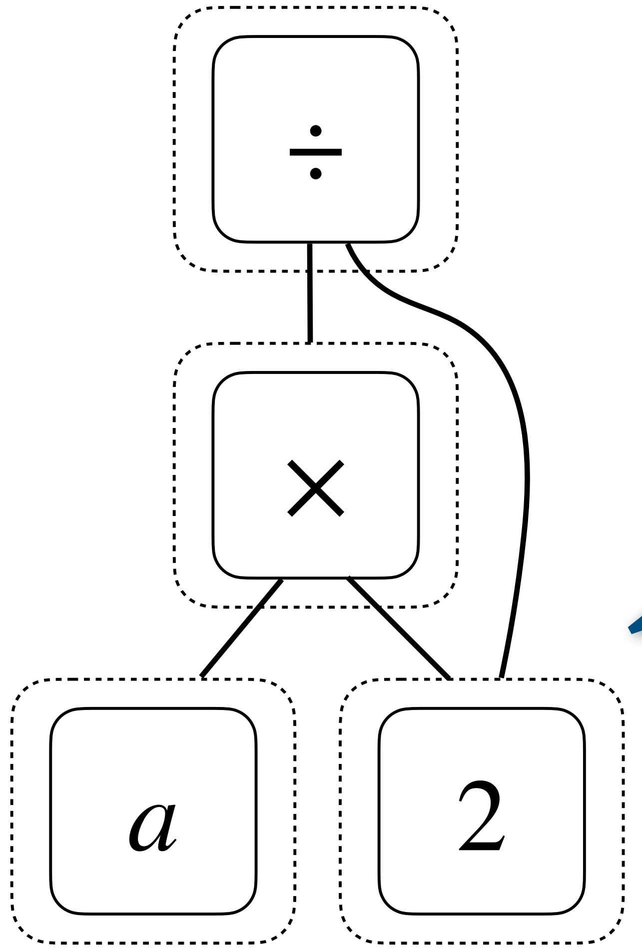

An E-graph

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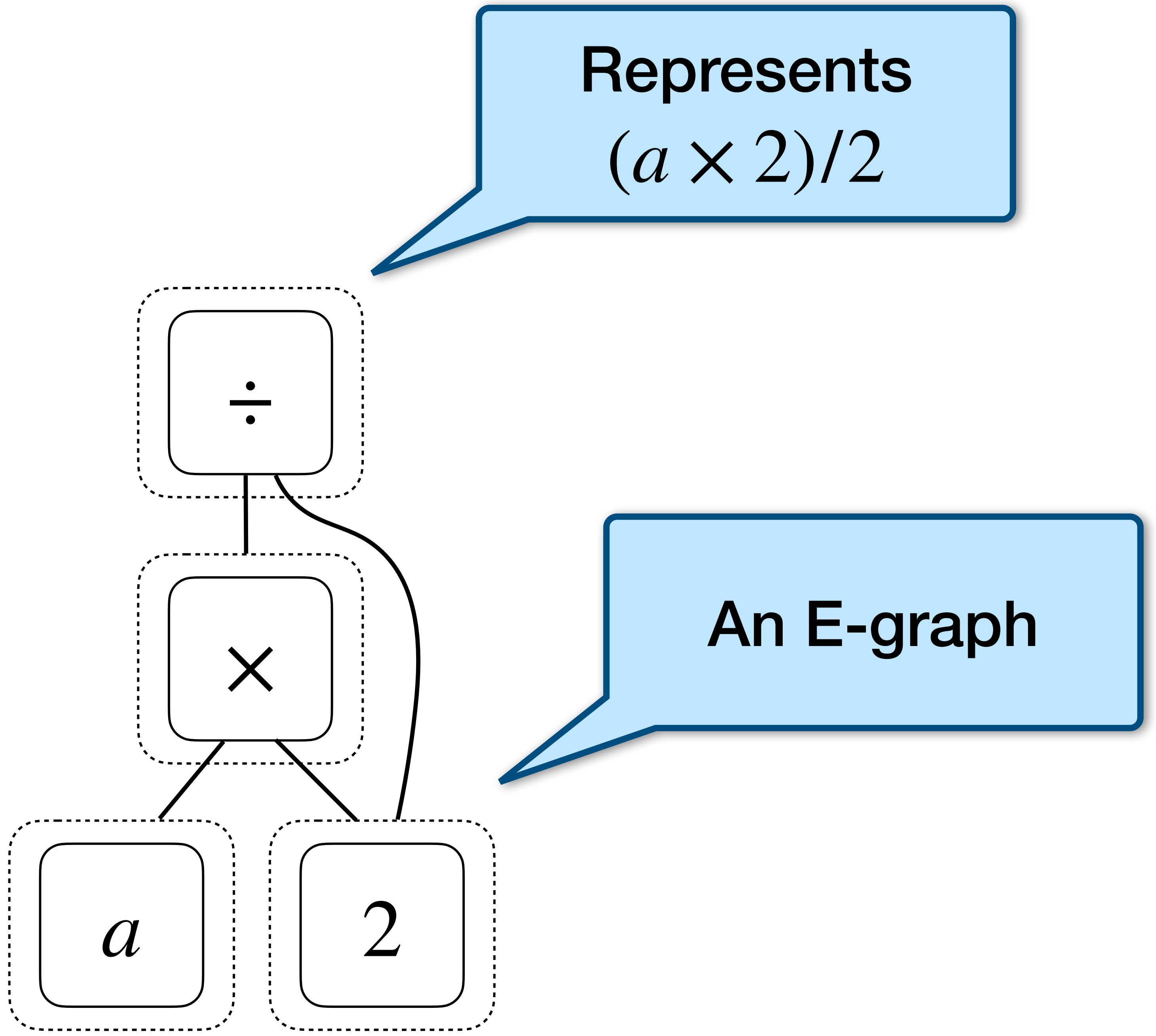

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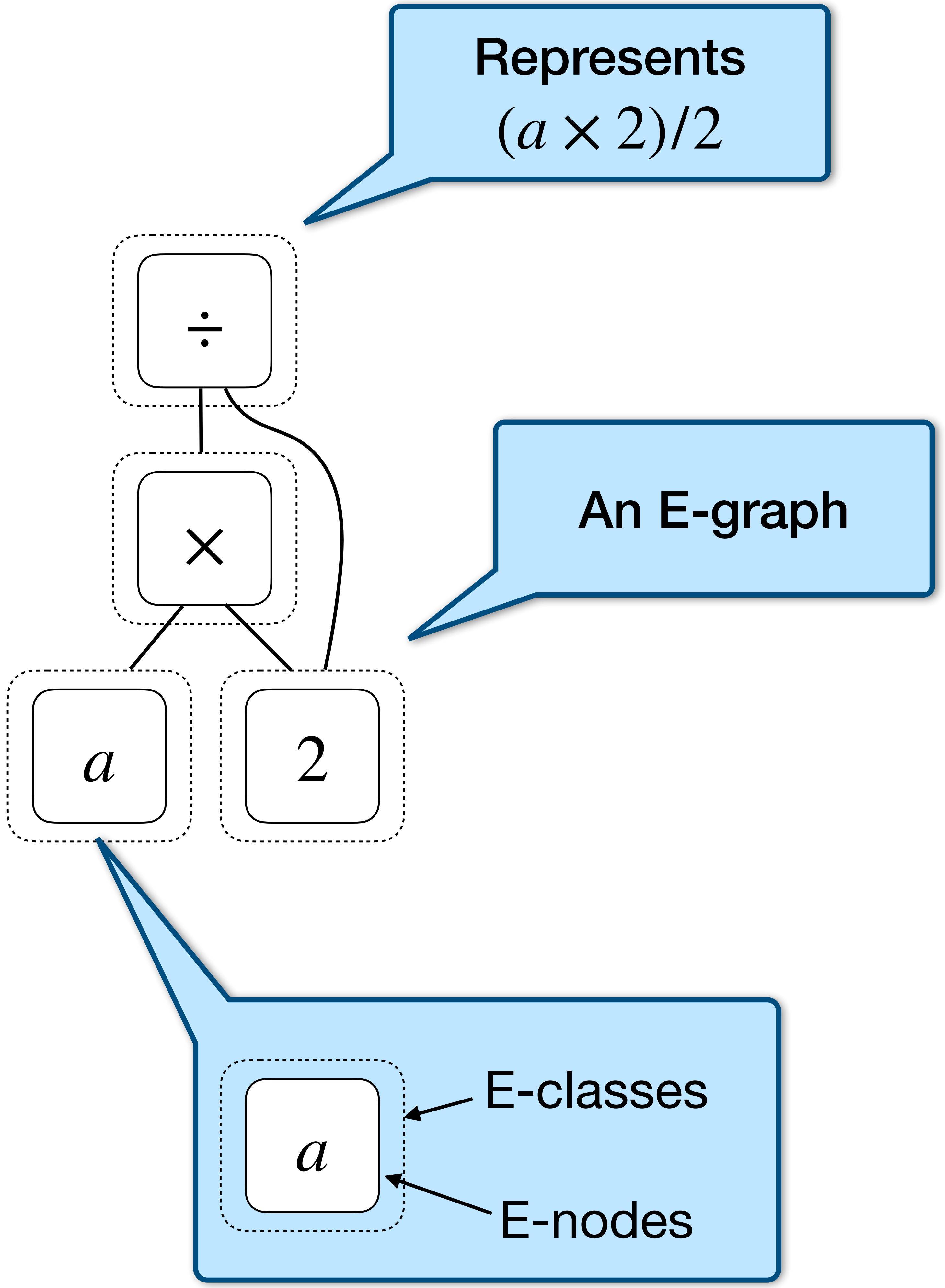

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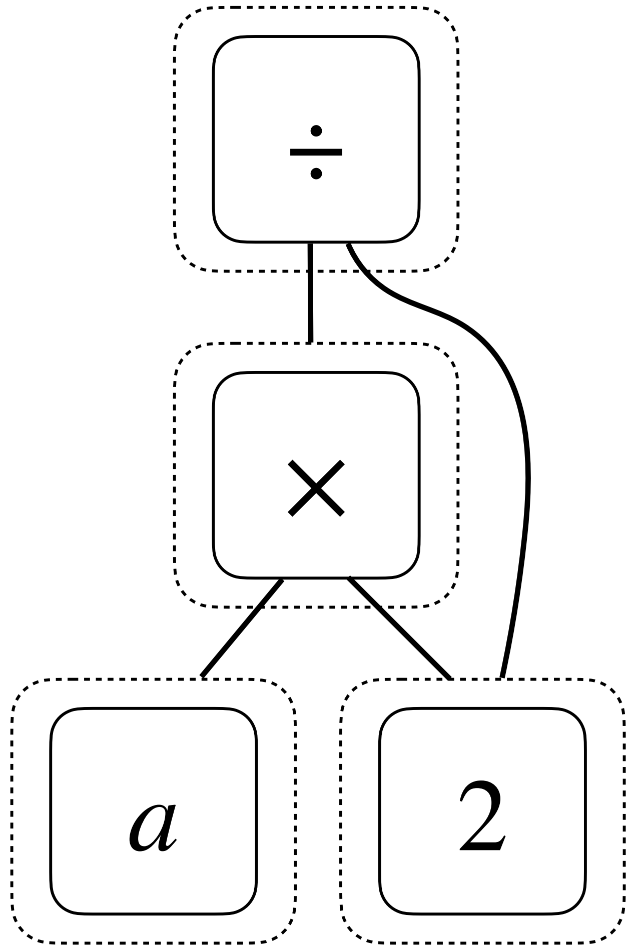

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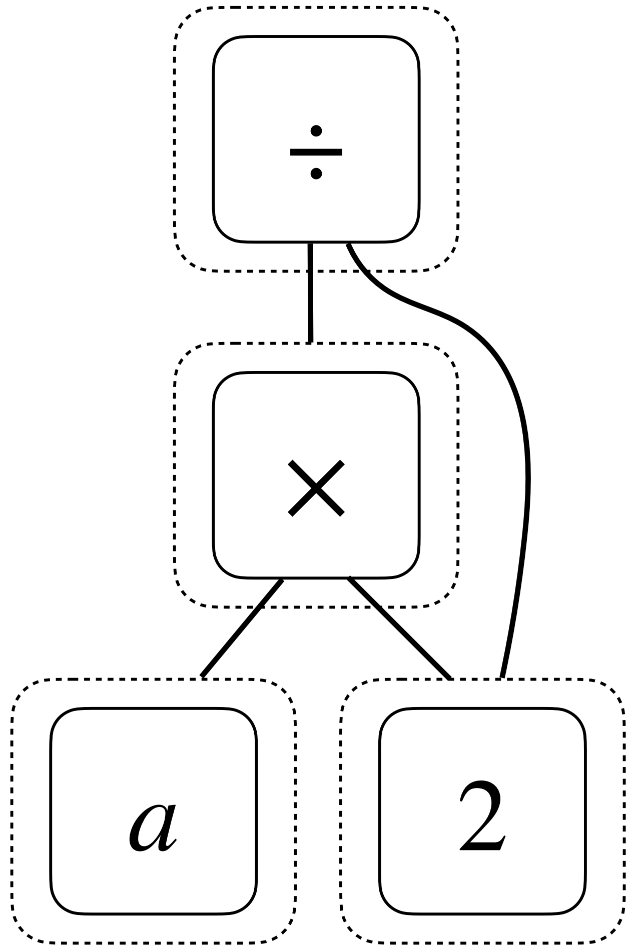

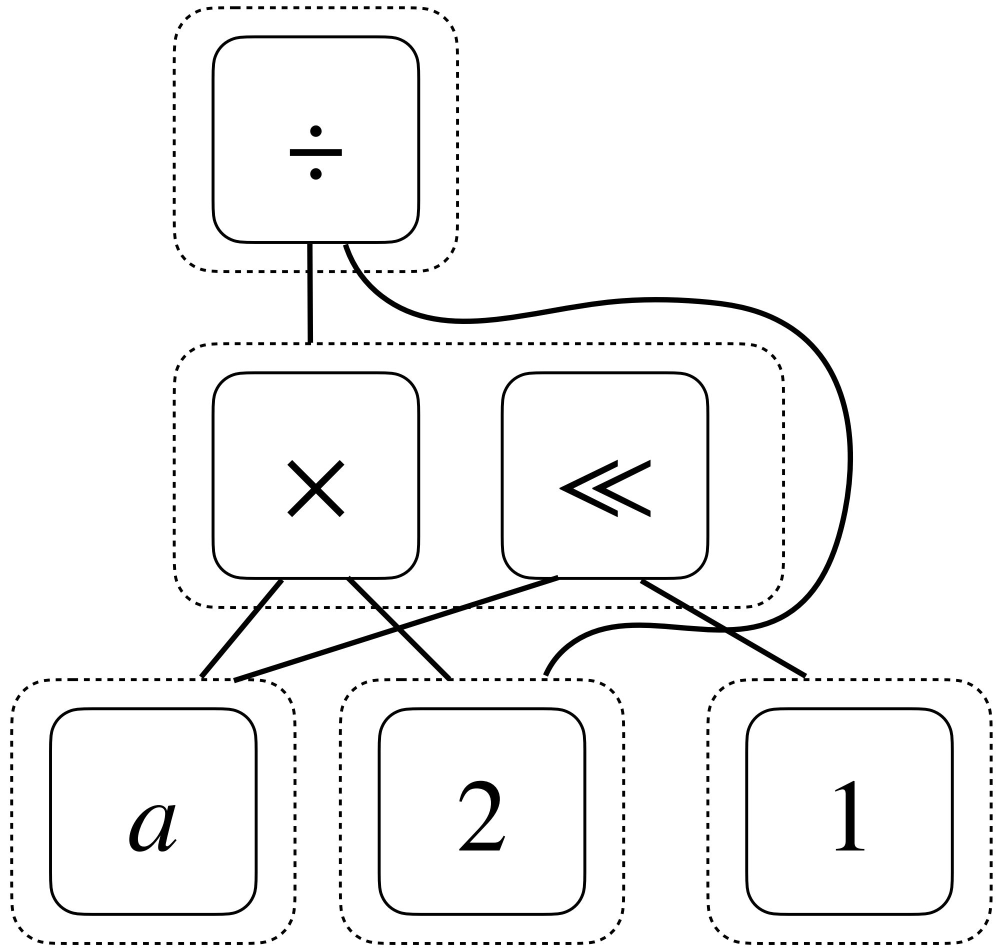


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x * 2 => x << 1
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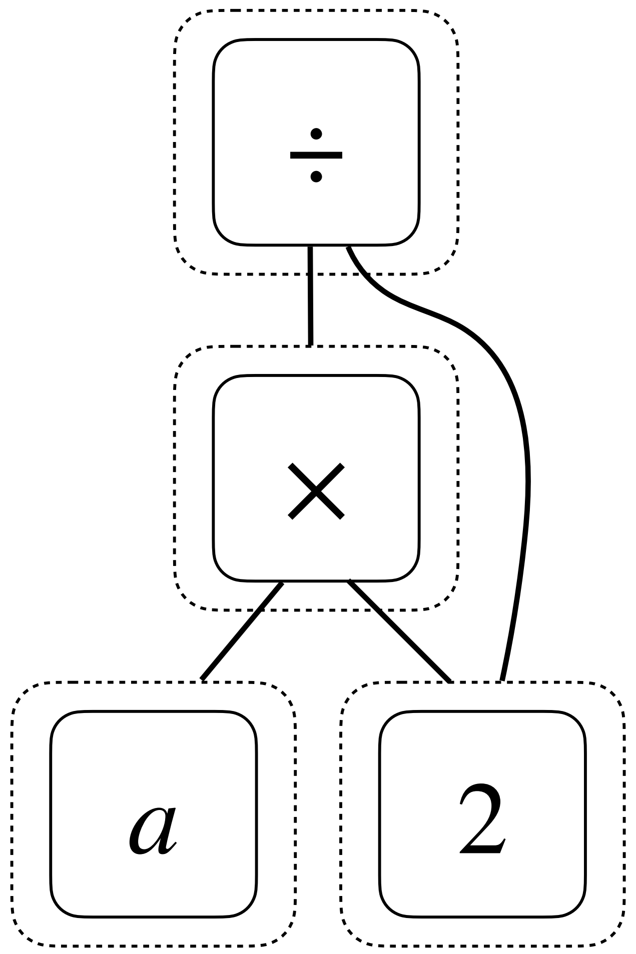

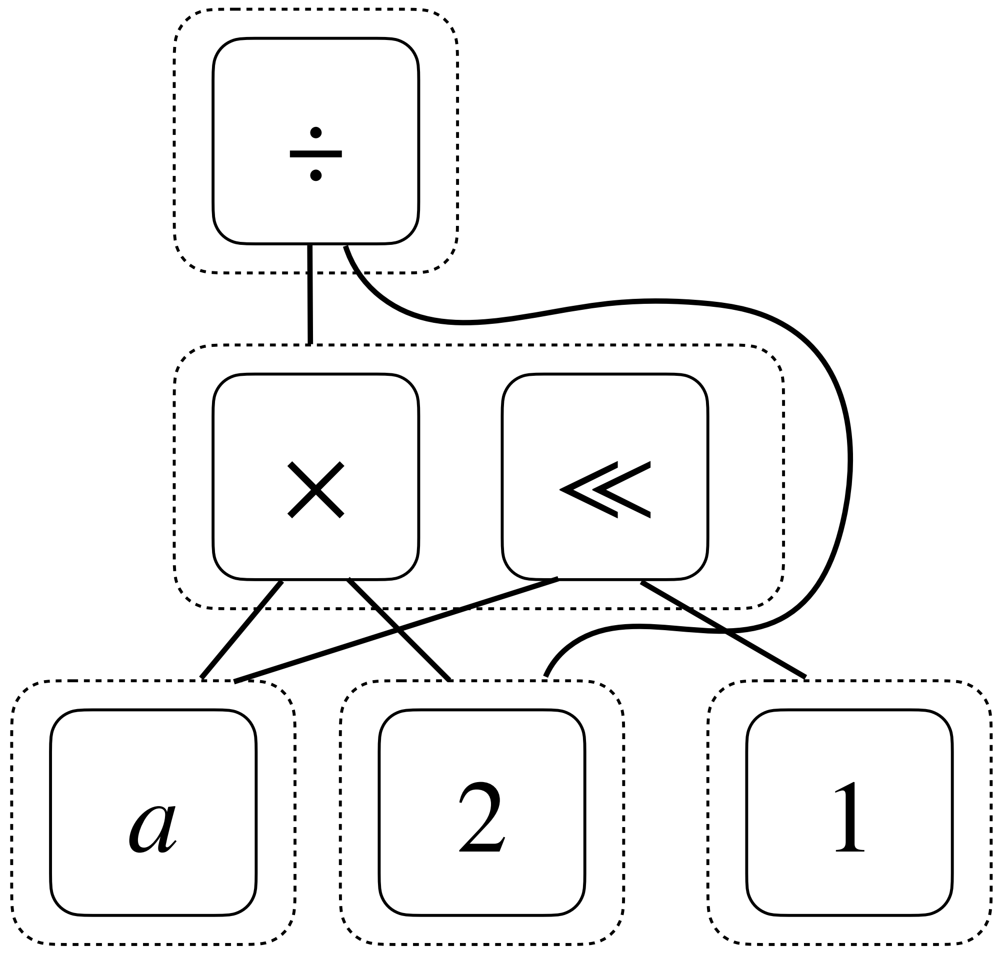

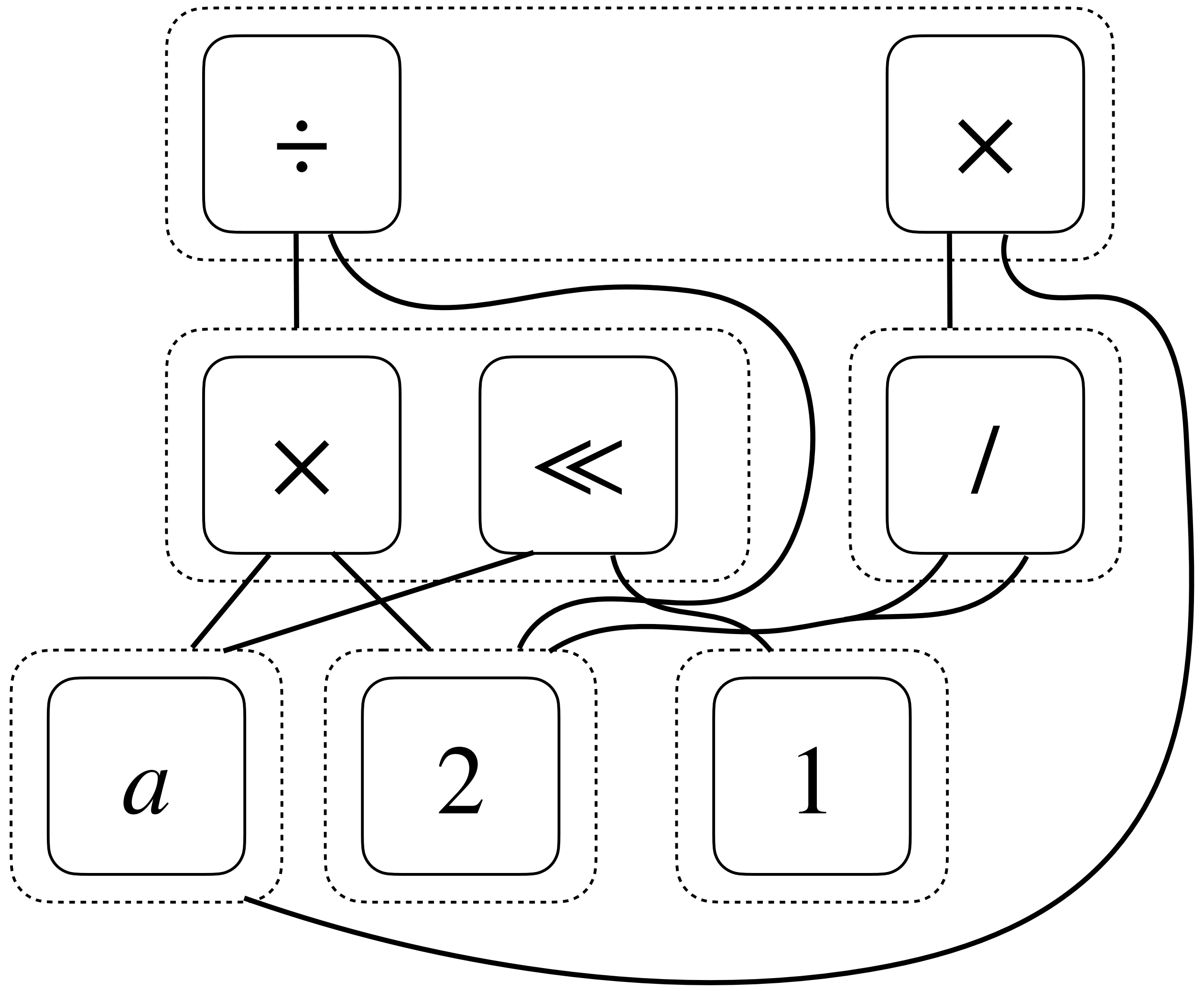

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(x * y) / z => x * (y / z)

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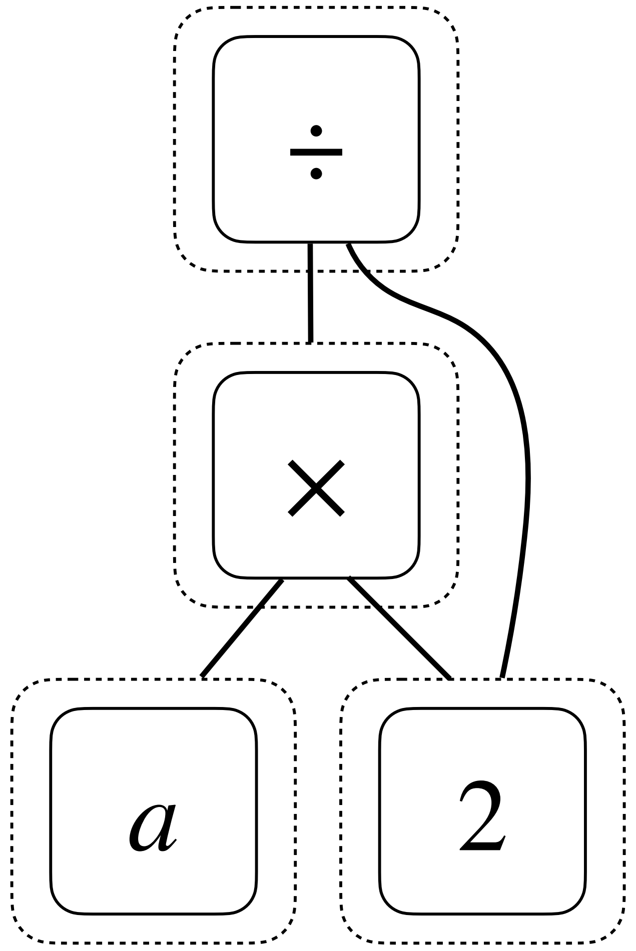

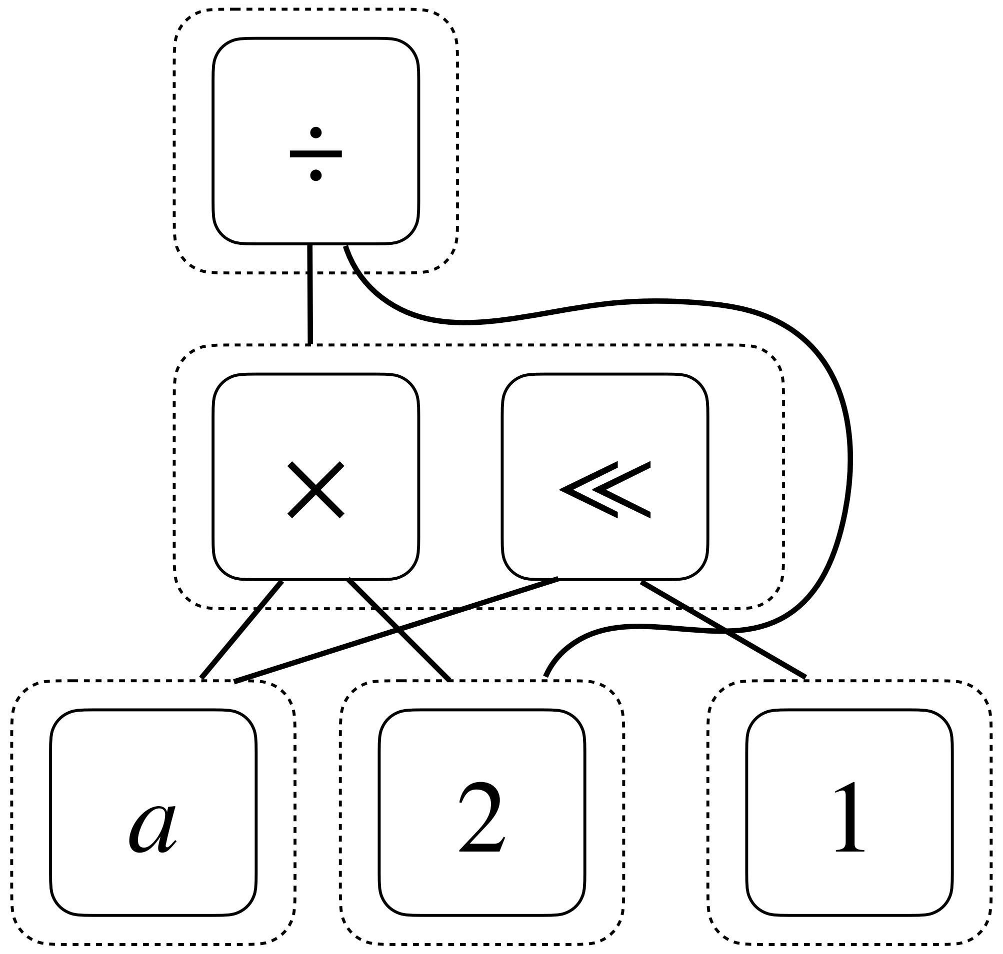

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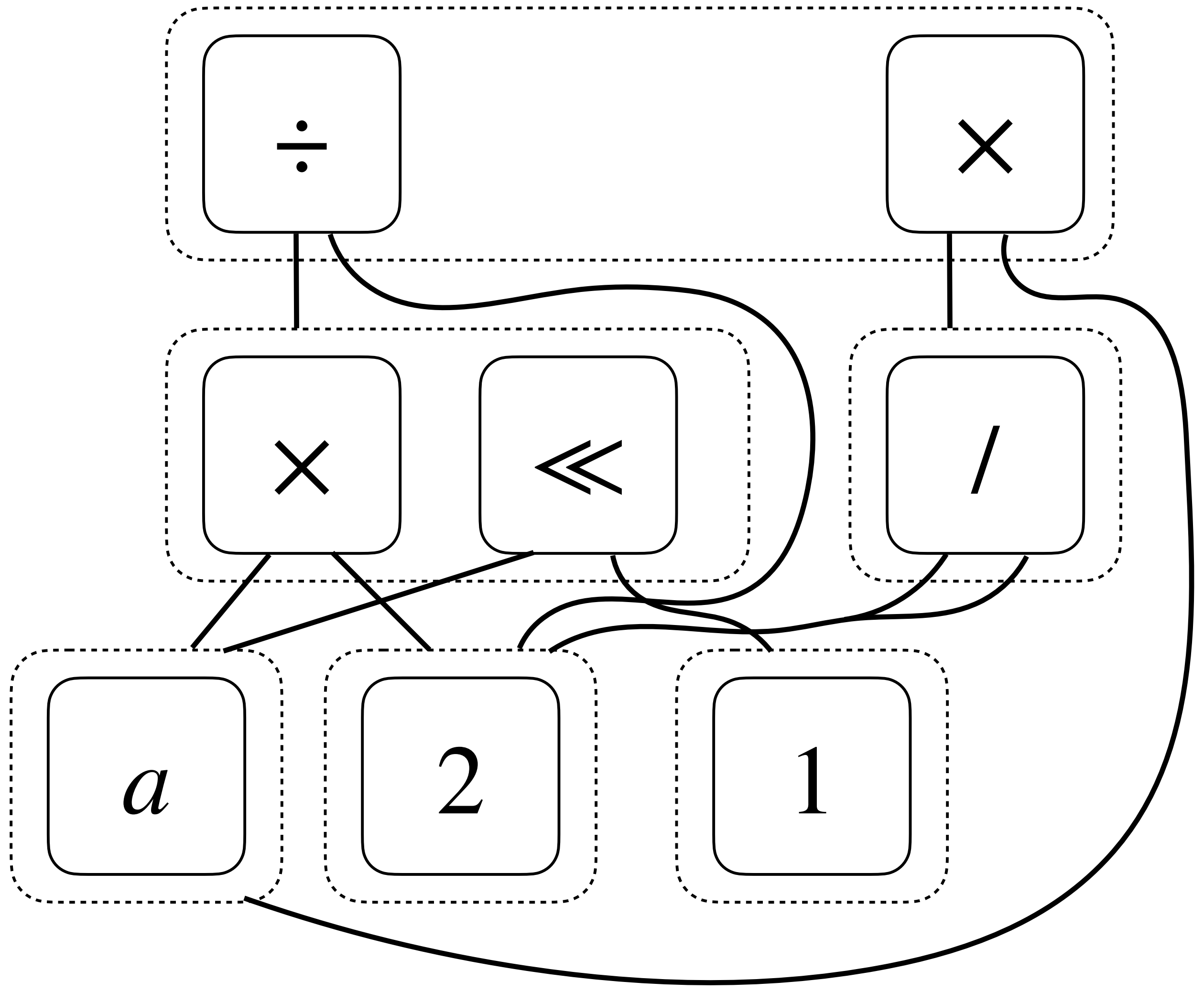

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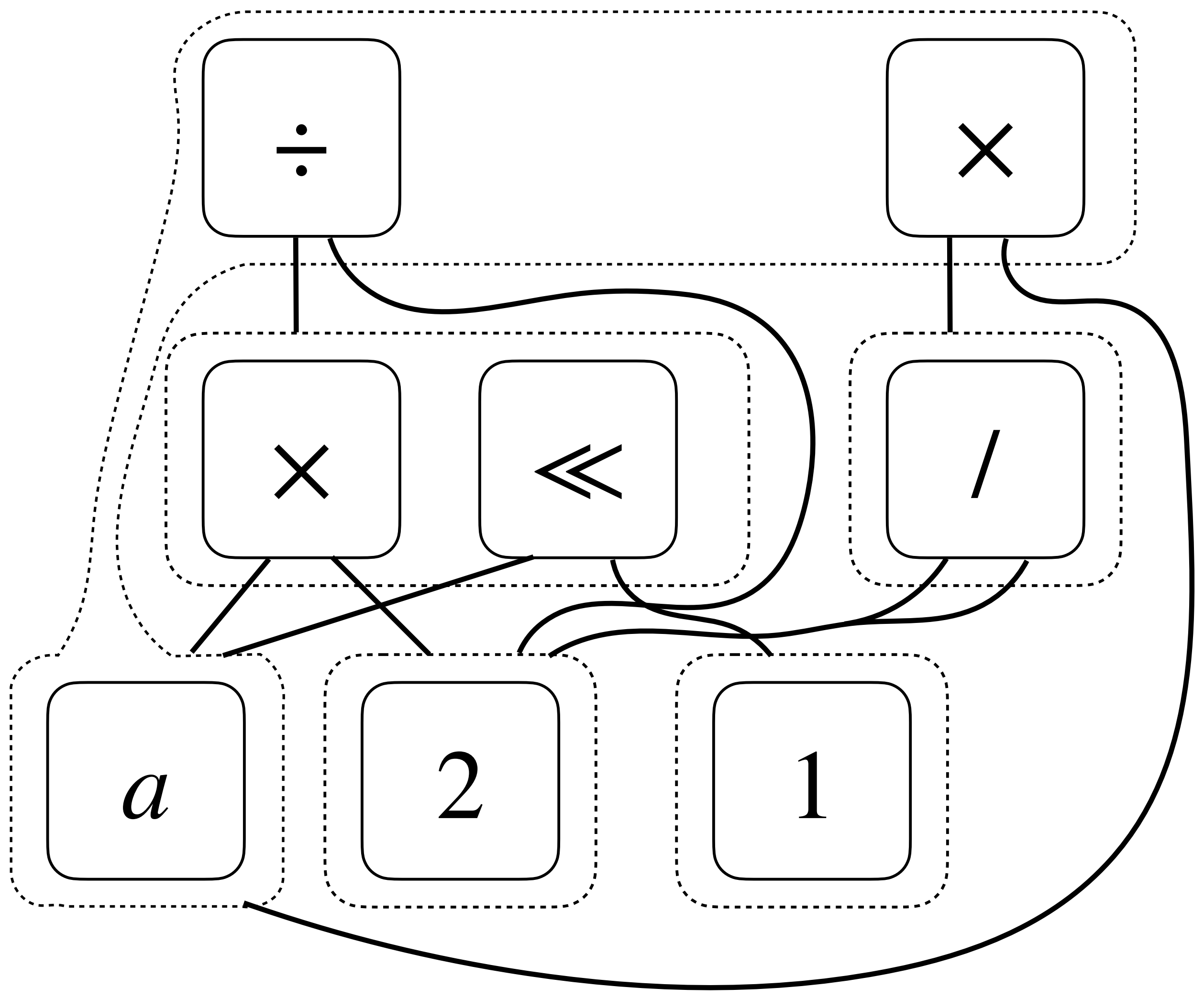

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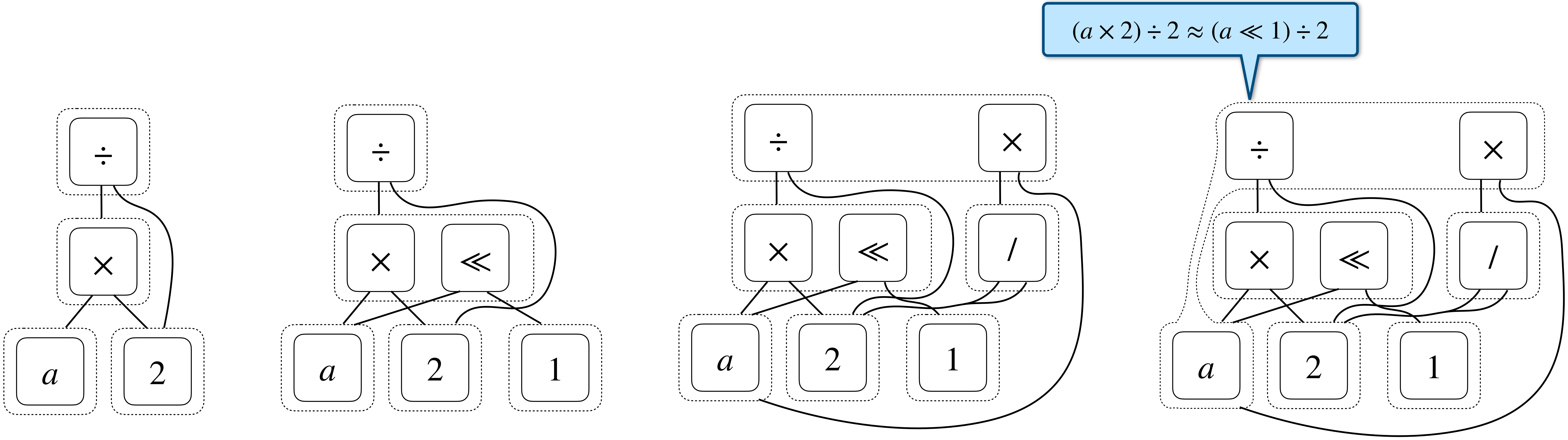

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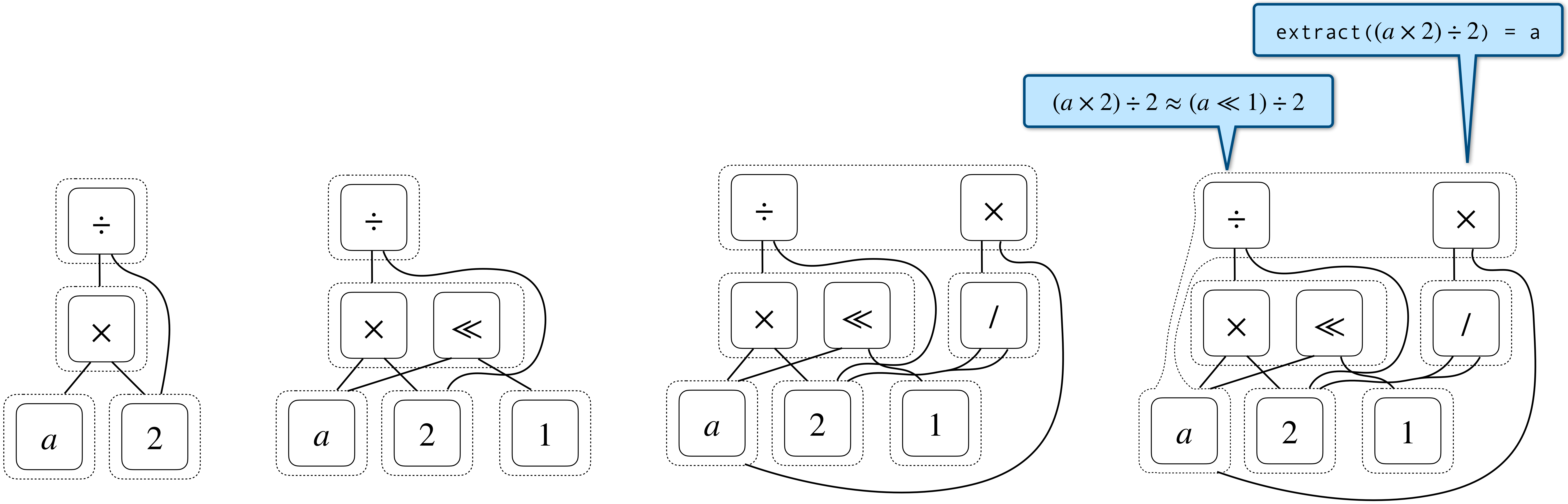

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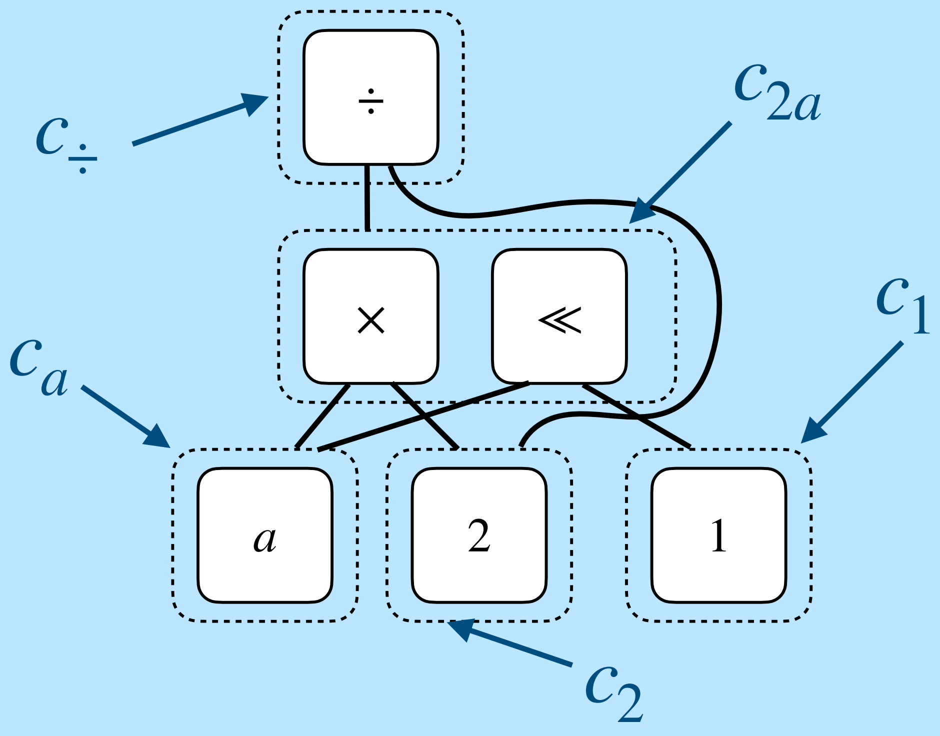

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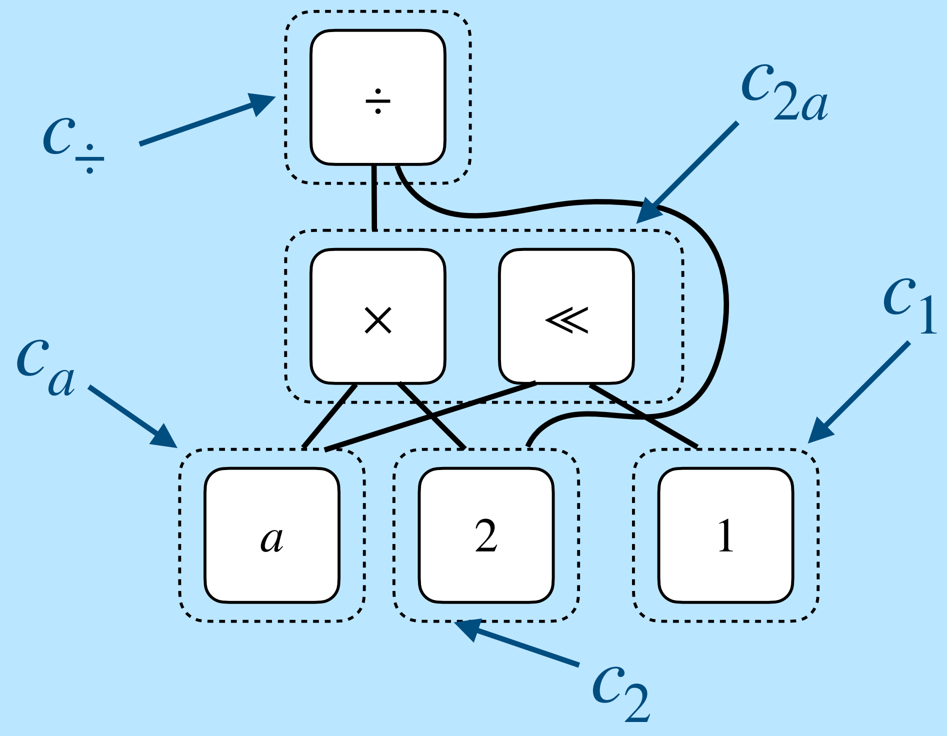

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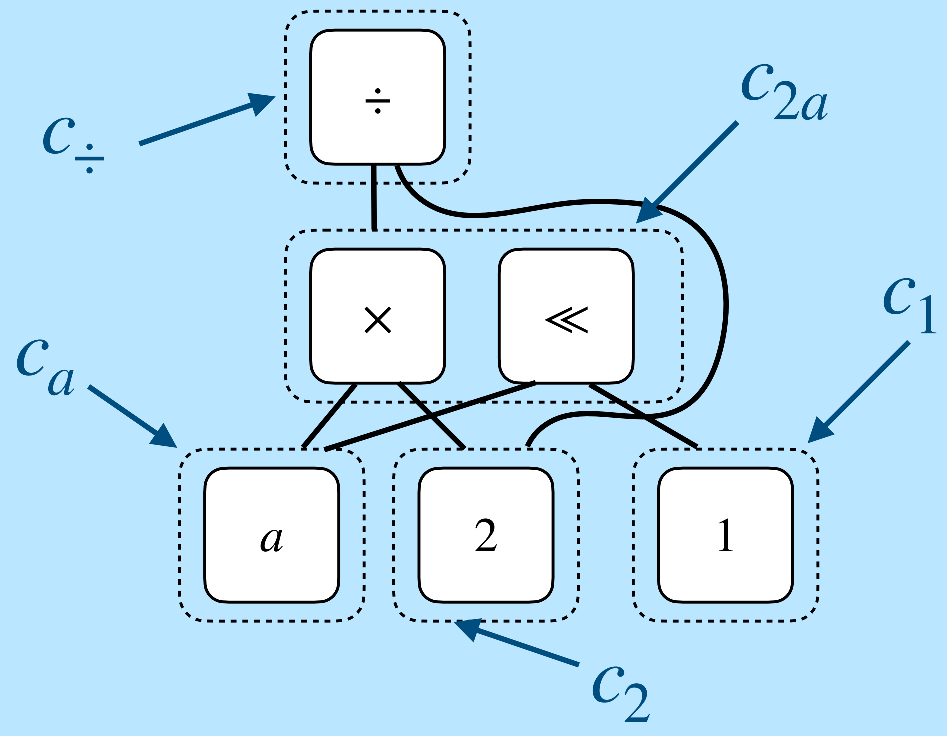

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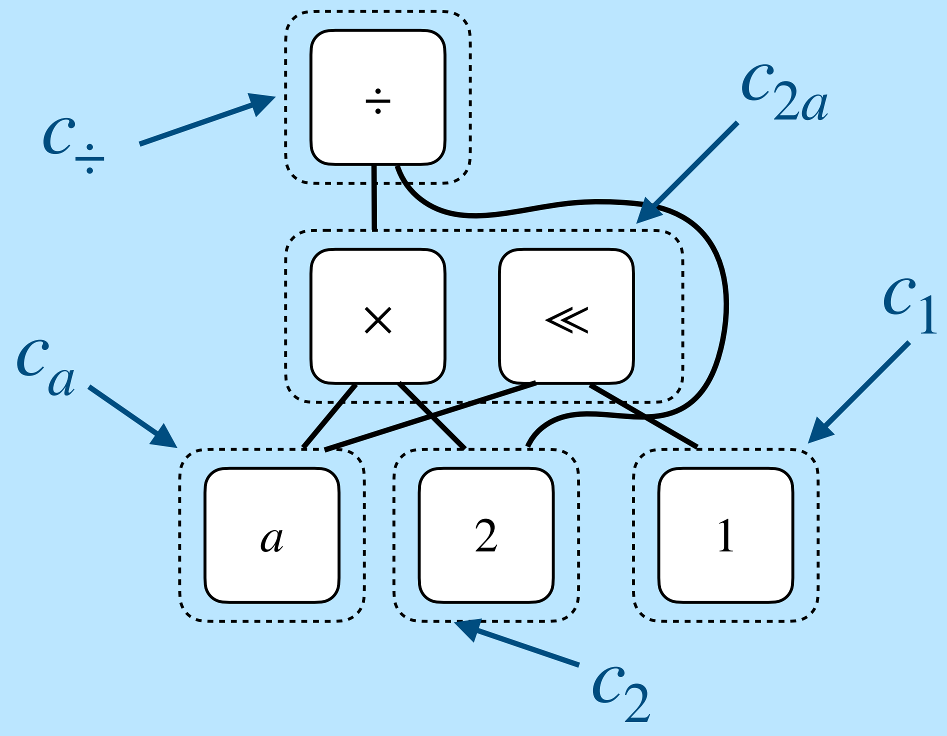

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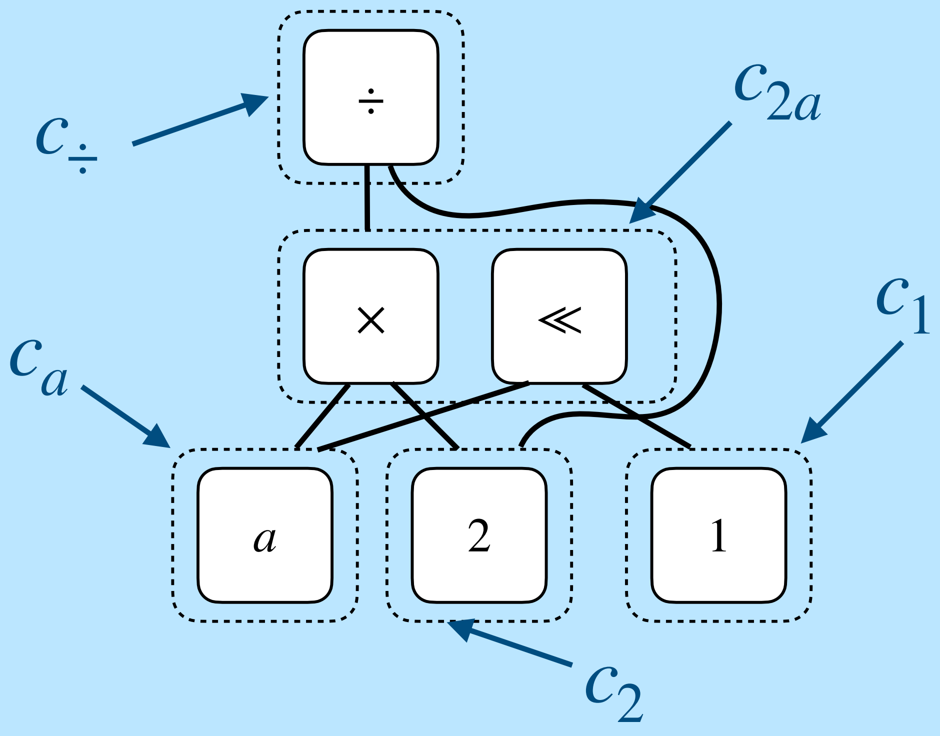

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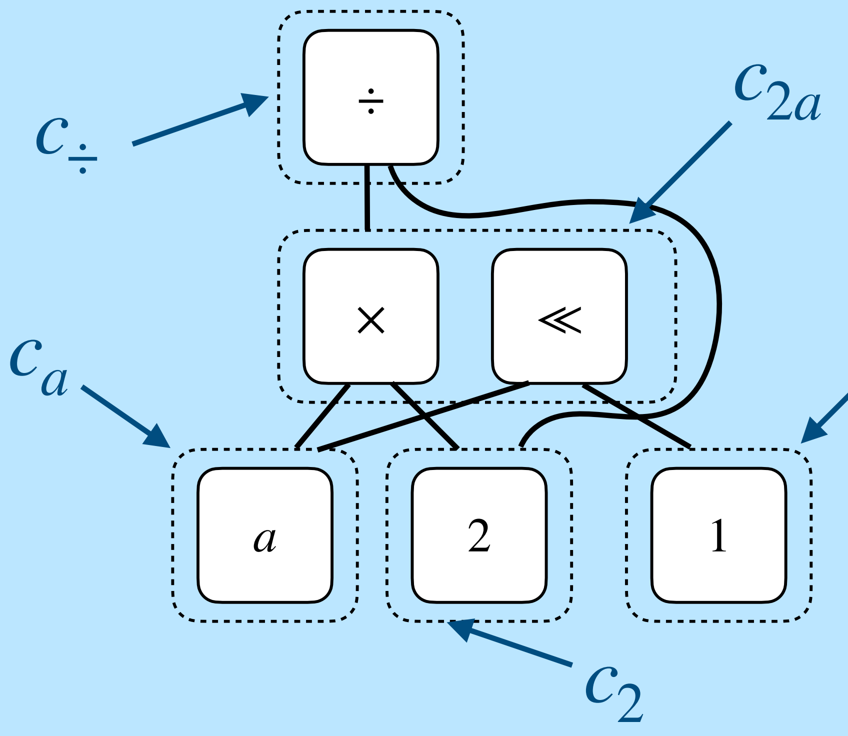

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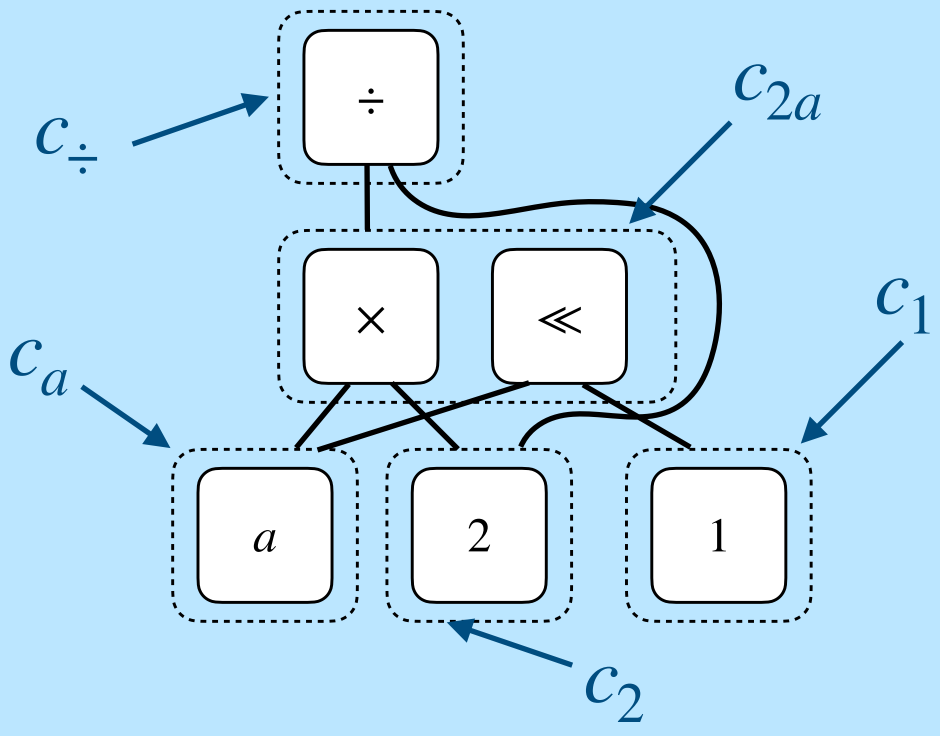

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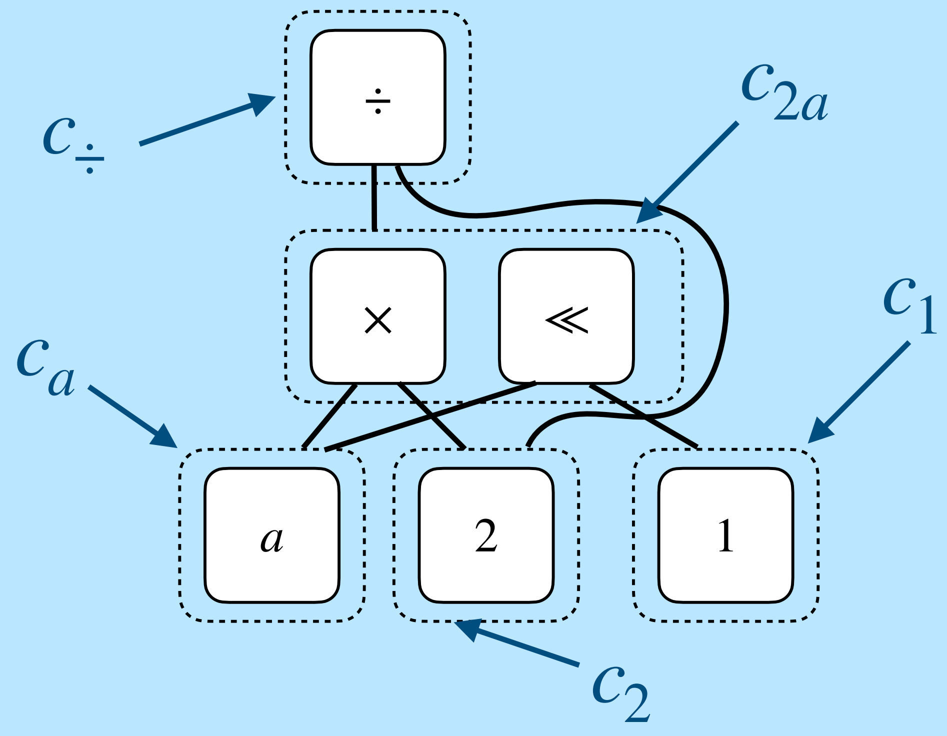

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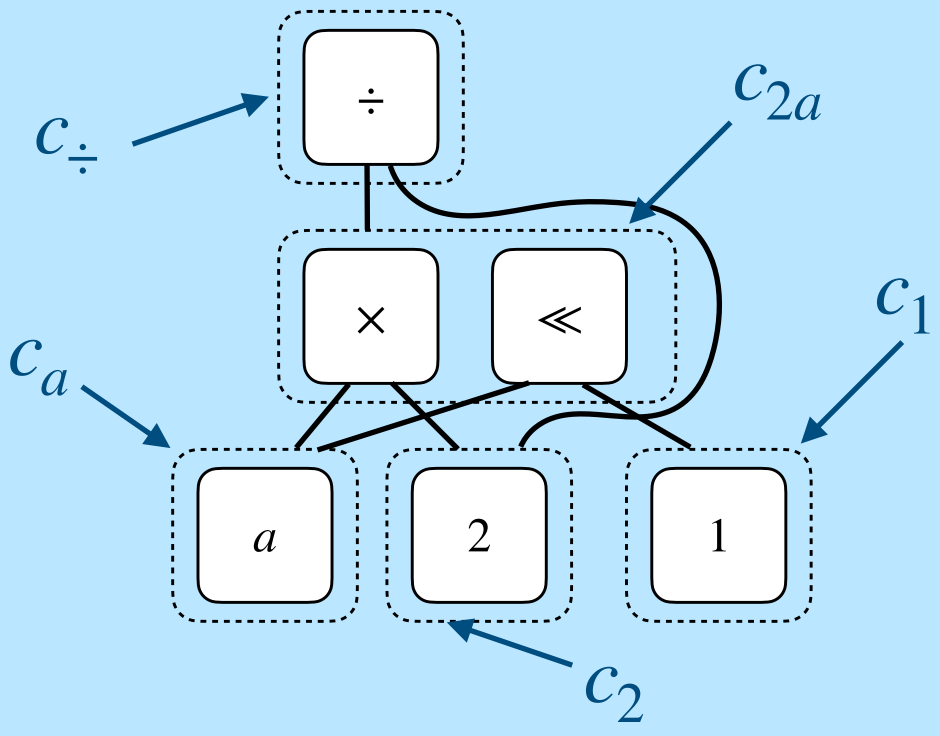

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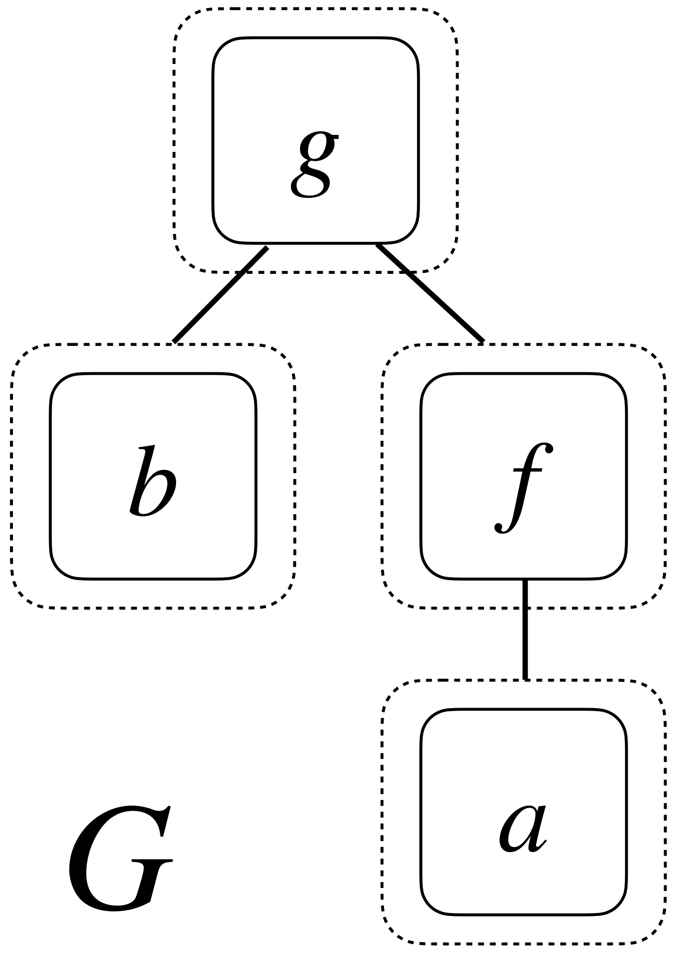

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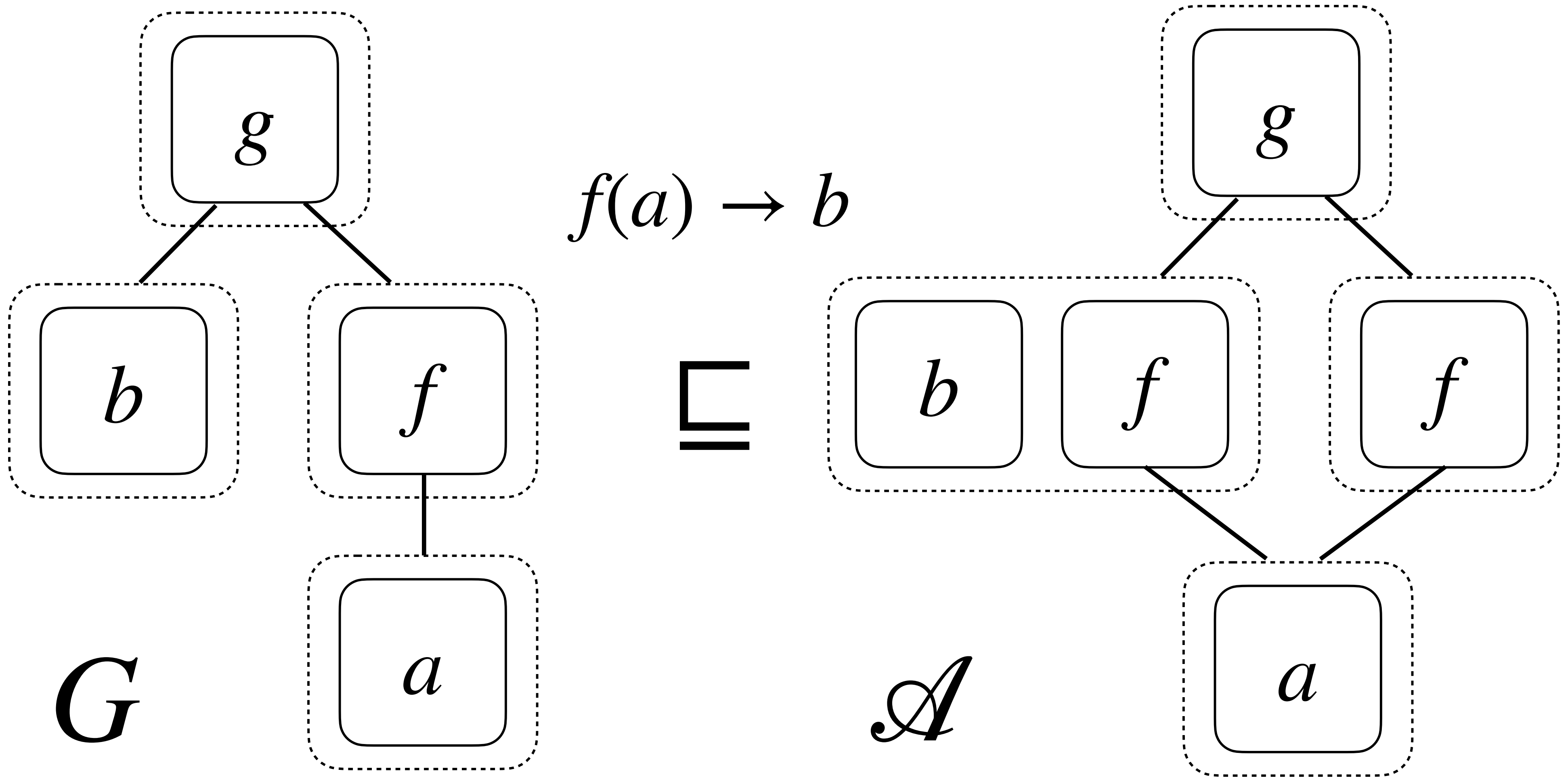

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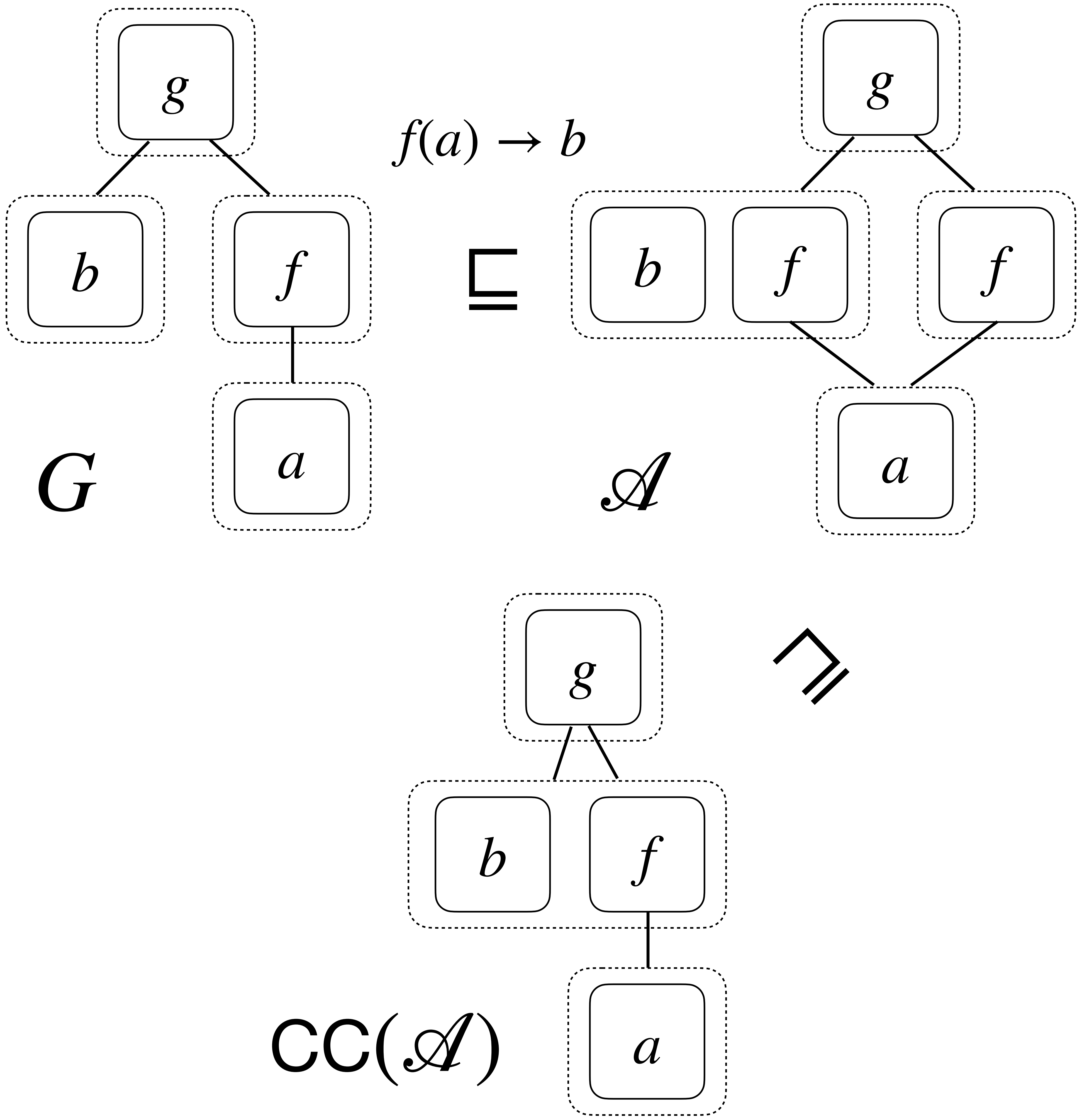

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- The *fixpoint semantics* of EqSat

$$\text{EqSat}(R, G) = \bigsqcup_{i \geq 0} \text{ICO}_R^{(i)}(G)$$

- The *model semantics* of EqSat: $\text{EqSat}(R, G)$ is a universal model of R and G
  (see the paper for defn. of universal models).
- The two semantics coincide.

Properties of EqSat

- (Inflationary) $G \sqsubseteq \text{EqSat}(R, G)$.
- (Finite convergence)
  If $\text{EqSat}(R, G)$ is finite, Equality Saturation converges in a finite number of steps.

EqSat and Term Rewriting

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- Let R be a TRS, $s \in T_\Sigma$, $G = \text{EqSat}(R, s)$
    - If $s \to_R^* t$, then $s \approx_G t$.
    - If $s \approx_G t$, then $s \leftrightarrow_R^* t$.

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  - If $s \approx_G t$, then $s \leftrightarrow_R^* t$.
- If R is bidirectional, $\to_R^*$, $\approx_G$, $\leftrightarrow_R^*$ coincide.
  - In this case, EqSat semi-decides the word problem.

Equality Saturation and the Chase

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- The Chase
  - A generalization of Datalog.
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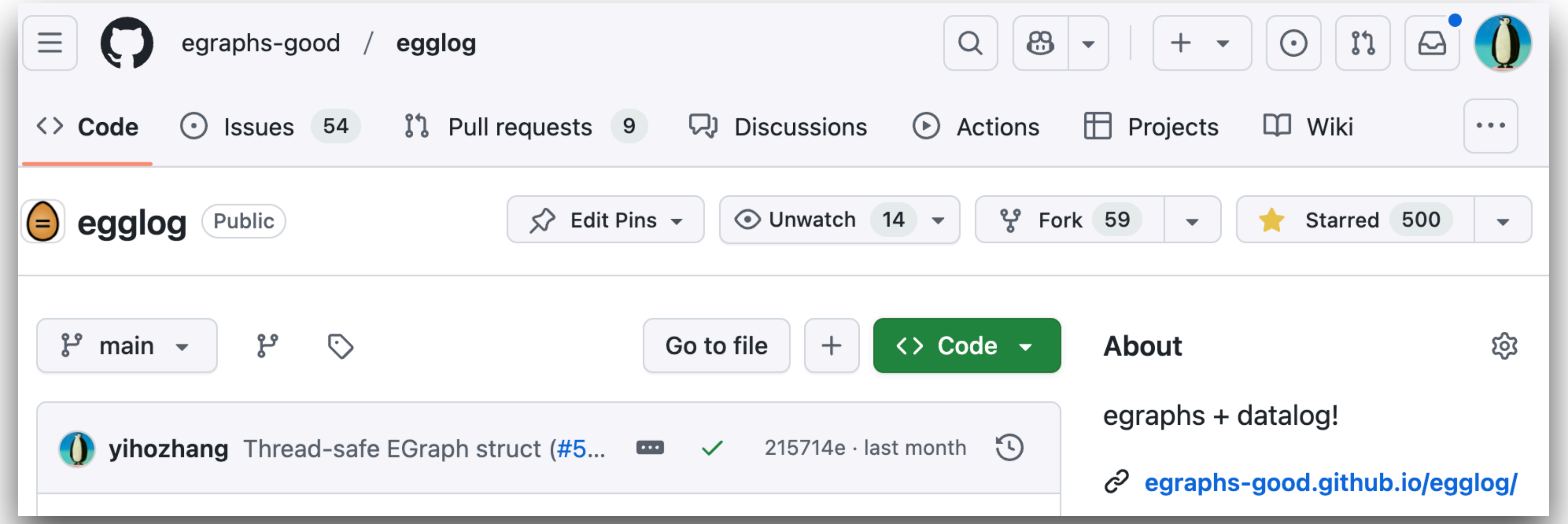

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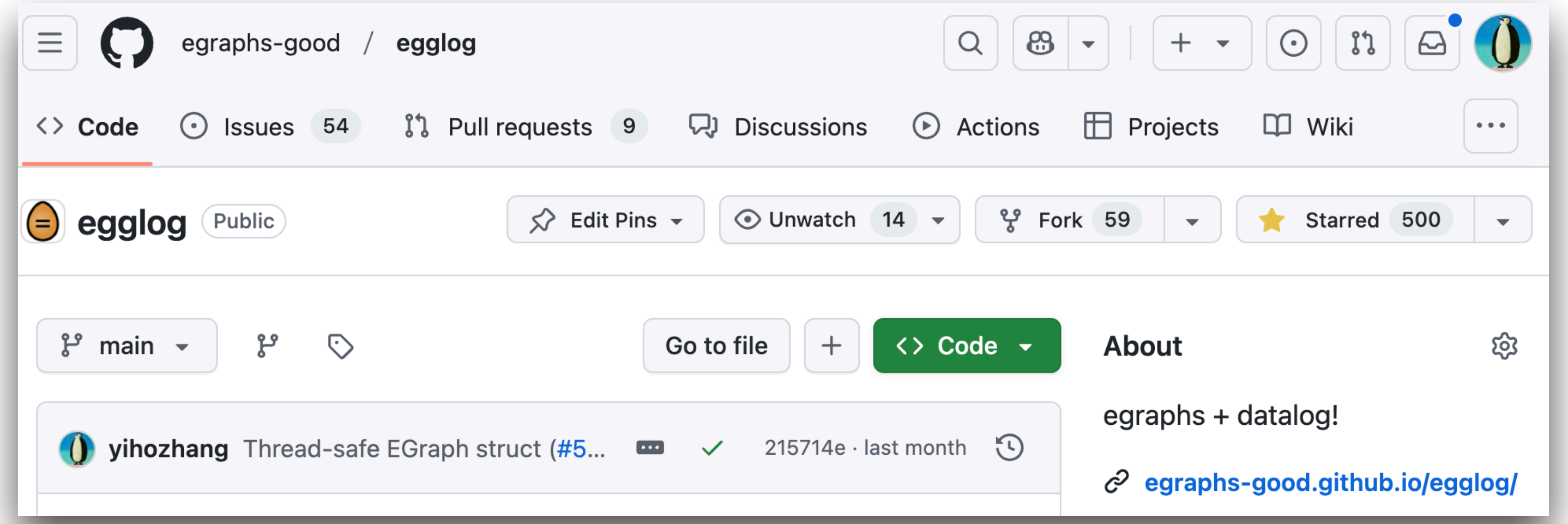

- Great speedup: Query optimization, semi-naive evaluation, worst-case optimal join, ...
- New expressive power: Multi-pattern rules, monotonic aggregation, ...

Termination Theorem

- (Single-instance) Does EqSat terminate with for a single term t?
  - Recursive enumerable (R.E.)—complete.
- (All-E-graph) Does EqSat terminate for all E-graph G?
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More details in the paper!

Summary

- The fixpoint and model semantics of Equality Saturation
- Connections to Term Rewriting and the Chase
- Undecidability of Termination
- Open problems
  - Extraction
  - Provenance

Also check out the egglog system!

GitHub: @egraphs-good/egglog