The E-graph extraction problem is NP-complete

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In this quick note, I will show that the E-graph extraction problem is NP-complete.

The E-graph extraction problem is defined as follows:

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 Input: An E-graph G and a cost function mapping E-nodes
        to positive numbers.
Output: A DAG t represented by G such that t has the
        lowest cost possible.
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First, the extraction problem is in NP[1] because it can be reduced to integer linear programming (ILP). Moreover, we show the extraction problem is NP-hard by reducing the minimum set cover problem to it.

The minimum set cover problem is defined as follows (adapted from Wikipedia):

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 Input: A set of elements {1, 2, ..., n} (called the universe)
        and a collection S of m sets whose union equals the
        universe.
Output: The smallest sub-collection of S whose union equals
        the universe.
```

We show an instance of how this minimum set cover problem can be reduced to E-graph extraction: consider the universe $U = \{1, 2, 3, 4, 5\}$ and the collection of sets $S = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}$. The smallest subset of S that covers all of the elements is { {1, 2, 3}, {4, 5} }.

Our construction is as follows:

1. For each $j \in U$, we create a corresponding E-class c_j.
2. For each collection S_i, we create an E-class c_{S_i} with a singleton E-node S_i.
3. For each $S_i = \{j_1, \dots, j_{l_m}\}$, we create a new E-node $u_{j_k}(c_{S_i})$ in E-class c_{j_k} for all j_k.

[1] Here we consider the optimization variant of the NP complexity class. The decision version of the extract problem is, given an E-graph and a cost function, does there exist a DAG represented by the E-graph with a given cost n?

4. We create a root E-class with a special E-node whose children include all C_{j_k}.
5. Every E-node has a uniform cost of 1.[2]

This will produce the following E-graph for our example.

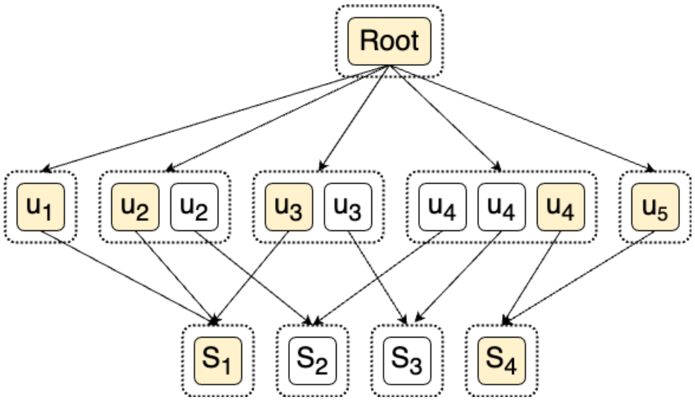

Figure 1: The E-graph from our example and the optimal extraction

The intuition behind this construction is that, to extract the root E-class, we have to cover all the elements in the universe, so we need to pick an E-node from each c_j. To cover all c_j's with the smallest cost means picking as fewer S_i E-nodes as possible, which corresponds to a minimum set cover.

As a side note, the construction here uses function symbols with non-constant arities (i.e., the root E-node). This can be fixed by replacing the root E-node with $O(n)$ many E-nodes with binary function symbols forming a depth $O(\log n)$ binary tree, so our reduction only requires unary and binary function symbols.

Update (Aug 29, 2023): Michael Stepp showed the NP-completeness of this problem via a similar reduction to Min-SAT in his thesis.

[2] In fact, the costs of the root E-node and u_{j_k}'s do not matter and can be set as zero, as these E-nodes will be in the extracted DAG anyway. We (arbitrarily) set their cost to be 1 (instead of say 0) to make sure the cost model is strictly monotonic.